Mathematical Reviews

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JUL 18 1960
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Vol. 21, No. 7 JULY–AUGUST, 1960 Reviews 4089–4876

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Reviews reprinted from Applied Mechanics Reviews, Referativnyĭ Žurnal, or Zentralblatt für Mathematik are identified in parentheses following the reviewer's name by AMR, RŽMat (or RŽMeh, RŽAstr. Geod.), Zbl, respectively.

MATHEMATICAL REVIEWS

Published monthly, except August, by

THE AMERICAN MATHEMATICAL SOCIETY, 190 Hope St., Providence 6, R.I.

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Editorial Office

MATHEMATICAL REVIEWS, 190 Hope St., Providence 6, R.I.

Subscription: Price $50 per year ($25 per year to individual members of sponsoring societies). Checks should be made payable to MATHEMATICAL REVIEWS. Subscriptions should be addressed to the American Mathematical Society, 190 Hope St., Providence 6, R.I.

The preparation of the reviews appearing in this publication is made possible by support provided by a grant from the National Science Foundation. The publication was initiated with funds granted by the Carnegie Corporation of New York, the Rockefeller Foundation, and the American Philosophical Society held at Philadelphia for Promoting Useful Knowledge. These organizations are not, however, the authors, owners, publishers or proprietors of the publication, and are not to be understood as approving by virtue of their grants any of the statements made or views expressed therein.

Printed in Great Britain by William Clowes and Sons, Limited, London and Beccles

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July-Dec
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Mathematical Reviews

Vol. 21, No. 7 July–August, 1960 Reviews 4089–4876

LOGIC AND FOUNDATIONS

4089:

★**Waismann, Friedrich. Introduction to mathematical thinking: The formation of concepts in modern mathematics.** With a foreword by K. Menger. Harper Torchbooks/The Science Library. Harper & Brothers, New York, 1959. xi+260 pp. Paperbound: $1.40.

The first edition [Frederick Ungar, New York, 1951] of this popular presentation, translated from the author's *Einführung in das mathematische Denken* [Gerold, Vienna, 1947], was reviewed in MR **13**, 899.

4090:

Yourgrau, Wolfgang; and Livingstone, Donald. On the nature of mathematical constants. Methodos **9** (1957), 3–29.

Mathematics is considered as a formal deductive system. Given is the comprehension of constant and variable, the difference between the concepts of constant and invariant, and various applications of these concepts. The views of the ancient Greeks (Parmenides, Pythagoras, Plato and others) are expounded. The question is raised of the role played by constancy in mathematics. As a basis for testing the validity of their point of view the authors take the transcendental idealism of Kant.

{Reviewer's remark. It is necessary to remark on (1) the Kantian position of the authors on the question of existence of mathematical concepts, (2) their incorrect estimate of Democritus as a pluralist who denies the reality of a sensible world and "less elevating and lofty than Parmenides".}

N. A. Kiseleva (RŽMat **1959** #7599)

4091:

★**Lewis, Clarence Irving; and Langford, Cooper Harold. Symbolic logic.** 2nd ed. Dover Publications, Inc., New York, 1959. ix+518 pp. Paperbound: $2.00.

Corrected and unabridged republication of the 1st ed. [Century, New York-London, 1932], with a brief Appendix III (by Lewis) on subsequent developments in the strict implication calculus $S2$ and the related systems $S1$–$S5$.

4092:

Curry, Haskell B. Calculuses and formal systems. Dialectica **12** (1958), 249–273. (German and French summaries)

Ein "Kalkül" besteht aus: (1) einem Alphabet $\mathfrak{A}$, d.h. einer endlichen Menge von Einzelzeichen, den $\mathfrak{A}$-Buchstaben (endliche Folgen von $\mathfrak{A}$-Buchstaben heißen $\mathfrak{A}$-Wörter), (2) einer Menge $\mathfrak{B}$ von weiteren Einzelzeichen, den Variablen ($\mathfrak{A} \cup \mathfrak{B}$-Wörter heißen $\mathfrak{A}$-Formeln), (3) einer endlichen Menge $\mathfrak{R}$ von Regeln der Form $A_1, A_2, \cdots, A_m \vdash A_0$ wobei die A_μ $\mathfrak{A}$-Formeln sind. Die Regeln definieren induktiv eine Klasse von $\mathfrak{A}$-Wörtern: die Klasse der $\mathfrak{R}$-Sätze. Im Anschluß an eine vom Referenten gebrauchte Erweiterung dieses Kalkülbegriffes definiert der Verf. einen "Kalkül vom Rang r" dadurch, daß $r+1$ Mengen $\mathfrak{B}_\rho$ $(\rho=0, 1, \cdots, r)$ von Variablen und $r+1$ Mengen $\mathfrak{R}_\rho$ von Regeln aus $\mathfrak{A} \cup \mathfrak{B}_0 \cup \mathfrak{B}_1 \cup \cdots \cup \mathfrak{B}_r$-Formeln gegeben sind. Für die Variablen aus $\mathfrak{B}_\rho$ dürfen jetzt nur die $\mathfrak{R}_\rho$-Sätze eingesetzt werden. Ein Kalkül vom Rang 1, in dessen Regeln keine Variablen aus $\mathfrak{B}_1$ auftreten, heißt ein L_1-Kalkül—auf diese lassen sich die obigen "Kalküle" und "Kalküle vom Rang r" leicht reduzieren. Ein Spezialfall sind die Kalküle vom Rang 0, die L_0-Kalküle genannt werden. Die Kalküle werden mit den "formalen Systemen" im Sinne des Verf. verglichen. Ein formales System enthält "Obs", die aus primitiven Obs (etwa dargestellt durch ein Alphabet $\mathfrak{A}$) durch Operationen (beliebigen Grades) erzeugt werden können. Verschiedene Erzeugungen ergeben per def. verschiedene Obs. Um alle Obs durch Zeichenfolgen darzustellen, untersucht der Verf. L_0-Kalküle mit einem Alphabet $\mathfrak{A}' \supset \mathfrak{A}$ und einer Menge $\mathfrak{R}_0$ von Regeln der speziellen Form $x_1, \cdots, x_m \vdash u_1 u_2 \cdots u_n$ wobei die x_μ Variable (für $\mathfrak{R}_0$-Sätze) und die u_ν $\mathfrak{A}'$-Buchstaben oder Variable sind, derart daß jedes x_μ mindestens einmal unter den u_ν vorkommt. Verf. beweist als Bedingung dafür, daß die Ableitungen von $\mathfrak{R}_0$-Sätzen isomorph zu den Erzeugungen von Obs eines formalen Systems sind ($\mathfrak{R}_0$ heißt dann tektonisch), eine Verallgemeinerung der Kriterien, mit denen—etwa für die Łukasiewicz-Notation—festgestellt wird, wann ein Ausdruck wohlgeformt ist. Der Unterschied zwischen Kalkülen und formalen Systemen (trotz prinzipieller Gleichwertigkeit) besteht darin, daß bei den Kalkülen von der Assoziativität der Verkettung von Zeichen inhaltlich Gebrauch gemacht wird. Die formalen Systeme sind insofern "more rigorous" und "put less emphasis on linguistic accidents".

P. Lorenzen (Kiel)

4093:

Harrop, Ronald. The finite model property and subsystems of classical propositional calculus. Z. Math. Logik Grundlagen Math. **5** (1959), 29–32.

Referring to a previous paper [Proc. Cambridge Philos. Soc. **54** (1958), 1–13; MR **20** #6] the author provides an example of a propositional calculus which is decidable and does not have the finite model property and which is a subsystem of the classical two-valued propositional calculus. The axiom schemes of the calculus are shown to be independent, and to possess instances no part of which is a tautology; and of each scheme it is shown that it is essentially used in some application of the single rule of modus ponens of the calculus.

P. C. Gilmore (Yorktown Heights, N.Y.)

4094:

Borkowski, Ludwik. Reduction of arithmetic to logic based on the theory of types without the axiom of infinity and the typical ambiguity of arithmetical constants. Studia Logica 8 (1958), 283–297. (Polish and Russian summaries)

The theory of types is extended by adding (a) variables $m, n, k, \cdots$ representing (numerical) quantifiers, (b) variables $f, g, h, \cdots$ and certain constants representing propositional functors with a quantifier as argument, and (c) a certain defined quantifier-forming functor with argument of the kind given in (b). We can then construct, for example, statements of the form "$\overset{k}{n}f(k)$", to be read "f holds for exactly n objects". Notions of "similarity", "cardinal number" and "natural number" are defined, and proofs are then sketched of versions of Peano's postulates and an axiom of infinity (in a form which asserts that for each natural number n there exists a property such that n natural numbers have this property). The formation-rules, axioms, rules of definition and proof, and the precise connection between the added symbolism and the type-theoretic formulas are all left rather vague.

E. Mendelson (New York, N.Y.)

4095:

Borkowski, Ludwik. On proper quantifiers. I. Studia Logica 8 (1958), 65–130. (Polish and Russian summaries)

Ω^n is called a quantifier of n arguments if the truth-value of the expression $\Omega_\alpha{}^n\Phi_1 \cdots \Phi_n$, where $\Phi_1, \cdots, \Phi_n$ are propositional expressions containing the variable α, is determined in each domain D by the cardinal numbers of the distribution classes of $\{\Phi_1, \cdots, \Phi_n\}$ in D. (Two objects b and c are said to belong to the same distribution class if, for each i, Φ_i has the same truth-value for b as for c. Clearly there are at most 2^n distribution classes.) Ω^n is a "proper" quantifier when the truth-value of $\Omega_\alpha{}^n\Phi_1 \cdots \Phi_n$ is determined in each domain D by the knowledge as to which distribution classes of $\{\Phi_1, \cdots, \Phi_n\}$ in D are empty or not. There are 2^{2^n-1} proper quantifiers Ω^n of n arguments. Any proper quantifier $\Omega_\alpha{}^n\Phi_1 \cdots \Phi_n$ can be defined by a formula in the predicate calculus built up from $\Phi_1, \cdots, \Phi_n$ by means of the propositional connectives and the existential quantifier $(\exists\alpha)$. Proper quantifiers of one and two arguments are studied in detail, and some results are obtained concerning those quantifiers which suffice for the definition of the other quantifiers.

E. Mendelson (New York, N.Y.)

4096:

Ono, Katuzi. A set theory founded on unique generating principle. Nagoya Math. J. **12** (1957), 151–159.

Axiomatizations of Zermelo-Fraenkel set-theory include axioms which assert the existence of certain basic sets and several axioms which allow us to generate new sets from existing ones. Here we have an axiomatic set-theory which turns out to be equivalent to Zermelo-Fraenkel set-theory, but has only one generating principle. There are however 3 primitive notions: the membership relation '$\in$', formation of the unit set '$\{\ \}$', and the constant 'p'. The axioms are: E1: $x \in \{x\}$; E2: $x \in \{y\} \rightarrow [\mathfrak{A}(x) \rightarrow \mathfrak{A}(y)]$; P1: $x \in p \rightarrow \{x\} \in p$; P2: $(x)(x \notin v) \rightarrow v \in p$; G: $(x)(Uy)\mathfrak{A}(x, y) \rightarrow (UEq)(y)[y \in q \rightarrow (Ex)(Eu)(x \subseteq u \in p \ \&\ \mathfrak{A}(x, y))]$, where we write '$(Ux)\mathfrak{A}(x)$' for '$(x)(y)[\mathfrak{A}(x) \ \&\ \mathfrak{A}(y) \rightarrow x = y]$', and '$(UEx)\mathfrak{A}(x)$' for '$[(Ux)\mathfrak{A}(x) \ \&\ (Ex)\mathfrak{A}(x)]$'. Axiom G is the only one which allows us to generate new sets.

L. N. Gál (New Haven, Conn.)

4097:

Shoenfield, J. R. On degrees of unsolvability. Ann. of Math. (2) **69** (1959), 644–653.

This paper continues the investigations of Kleene and Post, same Ann. **59** (1954), 379–407 [MR **15**, 772]; Spector, ibid. **64** (1956), 581–592 [MR **18**, 552]; Mučnik, Dokl. Akad. Nauk SSSR **108** (1956), 194–197 [MR **18**, 457]; and Friedberg, Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 236–238; J. Symb. Logic **22** (1957), 159–160 [MR **18**, 867; **20** #4488], on the semi-lattice of degrees of recursive unsolvability. The principal results (which in some cases we have stated in a slightly weakened form, in order to avoid complicated terminology) are the following. Theorem 1: There are degrees between $\mathfrak{n}$ and $\mathfrak{n}+1$ which cannot be represented by a set expressible in an $(n+1)$-quantifier form; in particular there are degrees < 1 which cannot be represented by recursively enumerable sets. Theorem 2: A function is recursive in a recursively enumerable set if and only if it has the form $f(x) = \lim_{n\to\infty} g(x, n)$ with g recursive (and likewise for higher steps of the Kleene hierarchy). Theorem 3: For every degree $\mathfrak{a}$ strictly between $\mathfrak{d}$ and $\mathfrak{d}'$, there is another degree incomparable with $\mathfrak{a}$ and also strictly between $\mathfrak{d}$ and $\mathfrak{d}'$. Theorem 4: $\mathfrak{a} = \mathfrak{b}'$ for some $\mathfrak{b}$ between $\mathfrak{d}$ and $\mathfrak{d}'$ if and only if $\mathfrak{a} \geq \mathfrak{d}'$ and $\mathfrak{a}$ is recursively enumerable in $\mathfrak{d}'$.

The author lists at the end some unsolved problems of which he says two are 'unlikely [to] be answered by any of the methods so far developed to deal with degrees'. In particular he asks whether the following propositions are true: (a) there exists an uncountable set of pairwise incomparable degrees; (b) there exists a linearly ordered set of degrees of power c. Neither of these questions is however about degrees at all; both are purely set-theoretic. Indeed (a) is true by a straightforward category argument; and (b) is evidently equivalent to the continuum hypothesis. The reviewer deplores the heavy computational methods, in the style of Kleene, used throughout this paper, and suggests that had appropriate set-theoretic arguments been used the author would not have permitted himself to state these two trivial 'unsolved problems'. Nonetheless, the results of the paper, especially Theorem 1, are of considerable interest to researchers in recursion theory.

J. Myhill (Berkeley, Calif.)

4098:

Mučnik, A. A. Isomorphism of systems of recursively enumerable sets with effective properties. Trudy Moskov. Mat. Obšč. **7** (1958), 407–412. (Russian)

This paper is a continuation of the reviewer's paper in Z. Math. Logik Grundlagen Math. **1** (1955), 97–108 [MR **17**, 118], the principal results of which are the following three: (I) two sets (of non-negative integers) are isomorphic if and only if they are one-one reducible to one another; (II) every recursively enumerable (r.e.) set is one-one reducible to every creative set, and hence (III) any two creative sets are isomorphic. The principal results of the paper under review are extensions of (I)—(III) to systems (i.e. finite sequences) of sets. Namely (I') two systems of k sets are isomorphic if and only if they are one-one reducible to one another; (II') every

system of k pairwise disjoint r.e. sets is one-one reducible to every system of k effectively inseparable r.e. sets, and hence (III′) any two systems of k pairwise disjoint effectively inseparable r.e. sets are isomorphic. (k is any number $\geqq 2$; we have for simplicity of statement strengthened I′ and weakened II′ in inessential respects.) Here $(\alpha_1, \cdots, \alpha_k)$ is called one-one reducible (isomorphic) to $(\beta_1, \cdots, \beta_k)$ if there is a one-one recursive function (a recursive permutation) f for which $\alpha_i = f^{-1}\beta_i$ $(i = 1, \cdots, k)$; and $\alpha_1, \cdots, \alpha_k$ are called effectively inseparable if $\prod \alpha_i$ is empty and if there is a recursive function g such that if $\prod (\alpha_i + \omega_{n_i})$ is empty then $g(n_1, \cdots, n_k)$ does not belong to $\sum (\alpha_i + \omega_{n_i})$. All these results and many others in the same area have been discovered since Mučnik's work (but in ignorance of it) by R. Smullyan in his doctoral dissertation [Princeton, 1959; to be published as *Theory of formal systems* in Annals of Mathematics Studies]. Though it is heartening to see the common direction of recursion-theoretic research in this country and the Soviet Union, it is sad that the lack of communication between the mathematicians of the two countries has led—now for the second time—to a needless duplication of effort in this area.

J. Myhill (Berkeley, Calif.)

4099:

★**Lacombe, Daniel. Sur les possibilités d'extension de la notion de fonction récursive aux fonctions d'une ou plusieurs variables réelles.** Le raisonnement en mathématiques et en sciences expérimentales, pp. 67–75. Colloques Internationaux du Centre National de la Recherche Scientifique, LXX. Editions du Centre National de la Recherche Scientifique, Paris, 1958. 140 pp. 1400 francs.

A semi-popular account, written for the general mathematician without background in logic or recursive functions, of the author's program of recursive analysis. We recommend it unconditionally to those who seek a rapid and painless initiation into this subject.

J. Myhill (Berkeley, Calif.)

4100:

★**Beth, E. W. Construction sémantique de la logique intuitioniste.** Le raisonnement en mathématiques et en sciences expérimentales, pp. 77–84. Colloques Internationaux du Centre National de la Recherche Scientifique, LXX. Editions du Centre National de la Recherche Scientifique, Paris, 1958. 140 pp. 1400 francs.

The author considers semantic entailments of the form $U_1, \cdots, U_m \vdash V_1, \cdots, V_n$ interpreted to mean that any model common to the statements $U_1, \cdots, U_m$ is a model for at least one of the statements $V_1, \cdots, V_n$. To establish such a statement in classical systems, it is sufficient to show that there exists no common model to $U_1, \cdots, U_m$ which is not a model of at least one V_i. Illustrations of techniques using both semantic tableaux and a system of formal rules to carry out such a proof are provided. The main body of the paper deals with the adaptation of these techniques to the proofs of semantic entailments in intuitionistic logic. This requires both stating formal rules of proof for semantic entailments, and defining the notion of an intuitionistic model in such a way that a semantic entailment between two sets of statements holds whenever the conjunction of the first set intuitionalistically implies the disjunction of the second. The semantic entailment is considered proved if the tableau constructed by means of the rules is closed; that the tableau is not closed demonstrates the unprovability of the entailment. The reviewer was unable to discover a concise definition of "closure" used in this sense.

E. J. Cogan (Bronxville, N.Y.)

4101:

★**Beth, E. W. Remarks on intuitionistic logic.** Constructivity in mathematics: Proceedings of the colloquium held at Amsterdam, 1957 (edited by A. Heyting), pp. 15–25. Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1959. viii + 297 pp. $8.00.

Discussion of objections raised by Kreisel and Gödel against the author's completeness proof for the intuitionistic first order predicate calculus [*Semantic construction of intuitionistic logic*, North-Holland, Amsterdam, 1956; MR **19**, 625]. The author adds various remarks on the formalization of intuitionistic arithmetic and on the application of his theory of semantic tableaux to this theory.

A. Heyting (Amsterdam)

4102:

★**Goodstein, R. L. Recursive analysis.** Constructivity in mathematics: Proceedings of the colloquium held at Amsterdam, 1957 (edited by A. Heyting), pp. 37–42. Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1959. viii + 297 pp. $8.00.

The author investigates the application of recursive methods in proofs of theorems similar to those of the form $\lim_{m\to\infty}(\lim_{n\to\infty} f(m, n)) = 0$, where $f(m, n)$ is a rational function of integral variables, noting that $\int_0^1 e^t dt = e - 1$ and $\sin \pi = 0$ are theorems of this form. Notions of recursive analysis defined and investigated are: relative equality, relative convergence, reduced sequence, relative continuity, relative derivative and integral. In defining these, a recursive relation $R(n, p)$ is said to hold for majorant n if there is a recursive function $M(p)$ such that $n \geqq M(p) \to R(n, p)$, and analogously for a ternary relation.

E. J. Cogan (Bronxville, N.Y.)

4103:

★**Lorenzen, Paul. Über die Begriffe "Beweis" und "Definition".** Constructivity in mathematics: Proceedings of the colloquium held at Amsterdam, 1957 (edited by A. Heyting), pp. 169–177. Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1959. viii + 297 pp. $8.00.

The paper contains clarifying remarks to the author's *Einführung in die operative Logik und Mathematik* [Springer, Berlin-Göttingen-Heidelberg, 1955; MR **17**, 223]. It is argued that, contrary to Aristoteles' theory of science, inductive definitions allow us to define new notions without starting from undefined notions, while the author's method of admissible (zulässig) formulas allows the deduction (in a certain sense) of formulas without starting from axioms. The difference between "zulässig" in the sense of effectively eliminable on the one hand, and in the classical sense on the other, is stressed.

A. Heyting (Amsterdam)

4104:

★**Myhill, John. Finitely representable functions.** Constructivity in mathematics: Proceedings of the colloquium held at Amsterdam, 1957 (edited by A. Heyting), pp. 195–207. Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1959. viii + 297 pp. $8.00.

The author considers a system consisting of the non-negative integers with a single function symbol f. The terms of the system are the integers and the values that f assigns to terms. Finite formulas are equalities, and sentences generated from these by the usual connectives. To explicate the notion of "a function effectively given" the author considers functions which can be described by an axiomatizable system of these formulas. Understanding intuitively that a standard model of such a system is one that assigns a number to every term, a function is said to be finitely representable if there is an axiomatizable system of finite formulas of which it is the unique standard model. Initial results establish the relation among the classical metalogical properties of completeness, decidability, axiomatizability, and so on, as they apply to sets of finite formulas. The two main theorems (5 and 8) provide alternative necessary and sufficient conditions that a function be finitely representable. The final theorem relates finitely representable functions to the Kleene hierarchy of recursive predicates. Finally, the author relates the result of theorem 5 to A. V. Kuznecov and B. A. Trahtenbrot, Dokl. Akad. Nauk SSSR **105** (1955), 897–900 [MR **17**, 1039], where the same result is obtained in a different context.

E. J. Cogan (Bronxville, N.Y.)

4105:

★**Rasiowa, H.; und Sikorski, R. Formalisierte intuitionistische elementare Theorien.** Constructivity in mathematics: Proceedings of the colloquium held at Amsterdam, 1957 (edited by A. Heyting), pp. 241–249. Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1959. viii + 297 pp. $8.00.

This is a very useful survey of results (by the authors, Mostowski, Tarski and others) on the algebraic and topological interpretation of the intuitionistic calculus of propositions and first order predicate calculus, and of extensions of the latter. *A. Heyting* (Amsterdam)

4106:

★**Sikorski, Roman. Der Heytingsche Prädikatenkalkül und metrische Räume.** Constructivity in mathematics: Proceedings of the colloquium held at Amsterdam, 1957 (edited by A. Heyting), pp. 250–253. Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1959. viii + 297 pp. $8.00.

Let $H(X)$ be the set of open subspaces of a topological space X. Mostowski [J. Symb. Logic **13** (1948), 204–207; MR **10**, 421] gave an interpretation of the logical signs by operations in $H(X)$, such that every deducible formula α of the intuitionistic first order predicate calculus P_1 is identically equal to X by the interpretation; this property is denoted by $\alpha_x = X$. Rasiowa and Sikorski [Fund. Math. **40** (1953), 62–95; MR **15**, 668] showed that, if $\alpha_x = X$ for every topological space, then α is deducible in P_1. A space X is called functionally free if $\alpha_x = X$ entails that α is deducible in P_1; X is called free if $H(X)$ contains a subalgebra isomorphic to the Lindenbaum algebra of P_1. Every free topological space is functionally free. Here it is shown that there exists a set X_0 of irrational numbers which is a free space. The proof is only sketched.

A. Heyting (Amsterdam)

4107:

Chomsky, Noam. On certain formal properties of grammars. Information and Control **2** (1959), 137–167.

The author here regards the grammar of a language L as a finite device capable of enumerating the sentences of L. A class of functions from which the grammar of L may be drawn may be the class of general unrestricted Turing machines, or the class of finite Markovian sources (finite automata). These are respectively the weakest and strongest conditions so far suggested for such grammar. The purpose of this paper is to investigate the relations among these two classes and two classes of strength intermediate to these. A grammar is defined formally as a system G which is a semigroup under concatenation with strings in a finite vocabulary in two parts, terminal and nonterminal. A two place relation (interpreted "can be written as") satisfies four axioms which permit derivations analogous to parsing of sentences. Three additional restrictions are provided, and the four grammars of types 0–3 are defined as those which satisfy no restriction, or one of the restrictions given. Type 0 grammars are Turing machines, type 3 grammars are finite automata, and types 1 and 2 are systems of phrase structure descriptions. The principal results of the paper are characterizations of languages of different types. It is shown that the class of grammars of type i ($i = 1, 2, 3$) is a proper subclass of type $i-1$. *E. J. Cogan* (Bronxville, N.Y.)

SET THEORY

See also 4401.

4108:

Kurepa, Georg. Ein Lemma über Transformationen. Math. Nachr. **19** (1958), 186–189.

Let S be a set; to each element x make correspond a subset $f(x)$ of the r-tuples of S so that each r-tuple contains x. Assume that the cardinal number of $f(x)$ is less than n for every x. Assume further that every r-tuple of S is contained in some $f(x)$. Then $r \leqq n$, $\aleph_0 \leqq n$ implies that the cardinal number of S does not exceed n. If $r = 2$ then the cardinal number of S does not exceed $2n^2$. Using these results the author formulates some statements equivalent to the continuum hypothesis.

P. Erdős (Budapest)

4109:

Oeconomidis, Nicolas. Sur les systèmes et les formes des ensembles. C. R. Acad. Sci. Paris **248** (1959), 2274–2276.

Dans cette Note, l'auteur a donné quelques définitions élémentaires sur les ensembles de points et leurs applications aux propositions simples dans la topologie élémentaire. D'après ses resultats, on voit que l'ensemble dérivé d'une somme $(\bigcup_{\gamma\in\Gamma} E_\gamma)'$ est contenu dans la somme des ensembles dérivés $\bigcup_{\gamma\in\Gamma} E_\gamma'$, si et seulement si le système $(E_\gamma : \gamma \in \Gamma)$ est fermé. *M. Kondô* (Tokyo)

4110:

★**Fraïssé, Roland. Sur une extension de la polyrelation et des parentés tirant son origine du calcul logique du $k^{ième}$ échelon.** Le raisonnement en mathématiques et en sciences expérimentales, pp. 45–50. Colloques Internationaux du Centre National de la Recherche Scientifique, LXX. Editions du Centre National de la Recherche Scientifique, Paris, 1958. 140 pp. 1400 francs.

Mise au point et généralisations de considérations antérieures de l'A. [Thèse, Paris, 1953; MR **15**, 296]. Une polyrelation [pl] est chaque suite finie de relations [rl] à plusieurs arguments. A chaque pl ou rl on attache son échelon k (entier ≥ 0) et sa signature: 0 est la seule signature d'échelon 0; une signature d'échelon 1 est une suite finie de 0's; celle d'échelon 2 est une suite de signatures d'échelons 0, 1, etc., chaque fois devant se présenter au moins une signature de l'échelon précédent. On définit plusieurs espèces d'équivalences ou parentés entre des pl, en particulier celle de $\frac{n}{i}$, $\mathscr{L}_k$, $\mathscr{L}$. Les classes d'équivalence par $\frac{n}{i}$ —classes Γ—font une suite dénombrable; l'A. en donne une énumération (Γ_j) bien déterminée "unicitaire". Une pl P est dite "déterminable" par une autre Q s'il existe une classe Γ contenant des QP', P' étant nécessairement isomorphe de P; si de plus $P'=P$, P est dite "isolée" par Q. P, Q étant pl d'échelons 2 et quelconque respectivement et de bases de puissances p, q avec $p \leq q$, alors P est dite e-déterminable par Q lorsqu'il existe un prolongement $\overline{P}$ de P sur une base $\overline{E}$ de puissance q telle que $\overline{P}$ soit déterminable par Q et que la relation $A(x)$ soit isolée par $\overline{P}$, $A(x)$ étant $+$ ou $-$ suivant que $x \in \overline{E}$ ou $x \notin \overline{E}$. "Pour que deux pl, P, Q de même signature donnent la même valeur à toute formule du k-ème échelon, il faut et il suffit que l'on ait $P \frac{n}{i} Q$ quel que soit n et quelle que soit la signature t, d'échelon au plus égal à k" ("Théorème fondamental"). Les démonstrations paraîtront ultérieurement. *Đ. Kurepa* (Princeton, N.J.)

4111:

★**de Possel, René; et Fraïssé, Roland. Hypothèses de la théorie des relations qui permettent d'associer, à un bon ordre d'un ensemble, un bon ordre, défini sans ambiguité, de l'ensemble de ses parties.** Le raisonnement en mathématiques et en sciences expérimentales, pp. 51–55. Colloques Internationaux du Centre National de la Recherche Scientifique, LXX. Editions du Centre National de la Recherche Scientifique, Paris, 1958. 140 pp. 1400 francs.

Dans l'article on considère en particulier les quatre problèmes: B (déterminer un ensemble de réels de puissance $\aleph_1$), C (bien ordonner l'ensemble R des réels), $C_{2^{\aleph_0}}$ (bien ordonner l'ensemble PR des parties de R), C_p (associer à un bon ordre d'un ensemble E de puissance p un bon ordre de PE). Les problèmes correspondants demandant l'unicité de la solution sont désignés par $\overline{B}$, $\overline{C}$, ···. L'on énonce ces deux hypothèses: T [resp. U]: Deux ordinaux dénombrables et L_2-parents [resp. L-parents] sont identiques. $U \Rightarrow \overline{B}$. Hypothèse H_p a pour but de résoudre $\overline{C}_p$ et à ce propos on considère un certain type T_P de relations dont la base est de puissance p, le type T_P étant déterminable par un ordinal. H_p entraînerait que le cardinal 2^p succède immédiatement à p. Si dans la description de H_p on remplace "déterminable" par "e-déterminable" on obtient "l'hypothèse K_p" entraînant la solution de $\overline{C}_p$. En particulier, $K_{\aleph_0} \Rightarrow \overline{C}$. *Đ. Kurepa* (Princeton, N.J.

COMBINATORIAL ANALYSIS

See also 4180, 4869.

4112:

de Bruijn, N. G. Generalization of Polya's fundamental theorem in enumerative combinatorial analysis. Nederl. Akad. Wetensch. Proc. Ser. A **62**=Indag. Math. **21** (1959), 59–69.

"Let D and R be finite sets, and G and H be groups acting on D and R, respectively. By F we denote the collection of all mappings of D into R. In F there is an equivalence notion induced by G and H. The elements $f_1 \in F$, $f_2 \in F$ are called equivalent if there exist elements $g \in G$, $h \in H$ such that $f_1(d)=hf_2(gd)$ for all $d \in D$. (gd denotes the effect of the permutation g on the element $d \in D$, so $gd \in D$, and hr is the effect of h on the element $r \in R$, so $hr \in R$.)"

The number of equivalence classes into which F is partitioned under this relation is obtained in terms of the cycle indices of the permutation groups G and H. This is a generalization of Polya's theorem [Acta Math. **68** (1937), 145–254], which is the special case in which H is the identity group of appropriate degree. The proof is analogous to that used by Polya. As special cases, a formula is given for enumerating the one-to-one mappings, and the case $D=R$ with simultaneous permutations is handled. Illustrative applications include the number (6) of types of coloring the sides of a square in the plane when there are three colors available and the number (7) of types of sets of vertices of a 3-cube which are equivalent to their complements. *F. Harary* (Ann Arbor, Mich.)

4113:

Auslander, L.; and Trent, H. M. Incidence matrices and linear graphs. J. Math. Mech. **8** (1959), 827–835.

The problem considered by the authors may be stated as follows. Let M be a matrix of residues mod 2 in which the columns are linearly independent. We seek necessary and sufficient conditions for M to represent a graph. In the representation rows are to correspond to edges and the columns are to determine a basis for the 1-cycles.

It is only necessary to consider the case $M=\left\{\begin{matrix} I \\ T \end{matrix}\right\}$, where I is a unit matrix. It is known that if M represents a graph then T represents a tree. The authors' conditions require first that the rows of T can be ordered so as to satisfy certain simple "incidence relations". Then a process is described, based on the deletion of a selected edge, whereby M is decomposed into two or more simpler matrices. It is shown that M represents a graph if and only if each of these simpler matrices does so.

W. T. Tutte (Toronto, Ont.)

4114:

Izbicki, Herbert. Unendliche Graphen endlichen Grades mit vorgegebenen Eigenschaften. Monatsh. Math. **63** (1959), 298–301.

In extension of his earlier work [Monatsh. Math. **61** (1957), 42–50; MR **19**, 161], the author proves the existence of graphs X with given automorphism group and the following properties: X is n-regular, $3 \leq n \leq 5$, k-chromatic, $2 \leq k \leq n$, and has connectivity c, $1 \leq c \leq n$. To the reviewer's knowledge the author is the first to point out the existence of uncountably many such graphs.

G. Sabidussi (New Orleans, La.)

4115:

Maghout, Khaled. Sur la détermination des nombres de stabilité et du nombre chromatique d'un graphe. C. R. Acad. Sci. Paris **248** (1959), 3522–3523.

"L'utilisation de l'algèbre de Boole permet une détermination simple des ensembles extérieurement stables minimaux, et des ensembles intérieurement stables maximaux, et d'arriver par ces derniers à déterminer le nombre chromatique d'un graphe." (Author's introduction.)

A set T of the points P of a digraph D is "out-stable" if every point $x \notin T$ has a directed line to a point of T. A set S of points of D is "in-stable" if no two points of S are adjacent in D. The out-stability [in-stability] number of D is the minimum [maximum] number of points in any such set T [S]. The chromatic number of D is the minimum number of in-stable sets whose union is P, i.e., the smallest number of colors which serve to color the points of D so that any two adjacent points have different colors.

To find the out-stability number of D, and indeed all the minimum out-stable sets, consider the points of D as boolean variables and form the conjunction C of the disjunctions of each point with all the points to which it is adjacent. Each term in the expression obtained from C by recasting C into disjunctive form gives a minimum out-stable set. The in-stability and chromatic numbers of D (which are symmetric concepts and do not depend on the directions of the lines of D) are found by analogous boolean algebraic procedures. These provide a considerable simplification of the algorithms contained in C. Berge, *Théorie des graphes et ses applications* [Dunod, Paris, 1958; MR **21** #1608].

F. Harary (Ann Arbor, Mich.)

4116:

Levine, Jack. On the application of MacMahon diagrams to certain problems in the multiplication of monomial symmetric functions. Duke Math. J. **26** (1959), 419–436.

MacMahon diagrams are the rectangular arrays introduced by P. A. MacMahon [*Combinatory analysis*, Vol. I, University Press, Cambridge, 1915; p. 237 et seq.] in connection with the expansion of products of monomial symmetric functions in terms of monomial symmetric functions. They arise from the partitioning of Hammond operators, and the enumeration of distinct arrays determines the numerical coefficients in the expansion. The author examines this enumeration problem, giving algorithms both for the case where the coefficient of a single term alone is required and for the case where the complete expansion is in question. These are applied to obtain general formulas for the products $(p^i)(q^j)$, $(p^iq^j)(r^k)$, $(p^iq^j)(r^ks^m)$, $(p^iq^jr^k)(s^m)$, $(p^iq^jr^k)(p^mq^n)$, $(p^i)(q^j)(r^k)$ and their special cases when some of the p, q, r, s are alike.

J. Riordan (New York, N.Y.)

ORDER, LATTICES

See also 4105, 4191, 4349, 4380.

4117:

Novotný, Miroslav. Über isotone Funktionale geordneter Mengen. Z. Math. Logik Grundlagen Math. **5** (1959), 9–28.

In this paper the author studies isotone functionals, that is, isotone functions from an ordered (=partially ordered) set to the set E of reals. He is motivated by an analogy with linear functionals on a vector space. Necessary and sufficient conditions are given in order that (i) an isotone functional on a set g be extendible to any superset (the functional be bounded), (ii) each isotone functional on each subset of a set g be extendible to all of g (each increasing chain and each anti-chain in each principal ideal, and each decreasing chain and each anti-chain in each dual principal ideal, be finite). For ordered sets g and G, let G^g be the cardinal product of g and G. Relations between E^g and E^h are obtained from knowledge of g and h, and conversely. For example, if x is an isotone mapping of g onto h, then there is a natural isomorphism of E^h onto a subset of E^g. Numerous other results pertaining to two-valued isotone functionals and cardinal product are presented.

S. Ginsburg (Gardena, Calif.)

4118:

Schmidt, Jürgen. Zu einem Axiom von Baer in Verbänden. Arch. Math. **10** (1959), 104–108.

In the present paper the author studies a condition for complete lattices introduced by Baer [Trans. Amer. Math. Soc. **64** (1948), 519–551; MR **10**, 425] and investigated by the reviewer [Indag. Math. **18** (1956), 435–443; **19** (1957), 182–189, 190–195, 220–226; MR **18**, 186; **19**, 240, 243] and by the reviewer and J. de Groot [Indag. Math. **18** (1956), 596–601; MR **18**, 461]. A complete lattice L satisfies the condition of Baer if for each system of elements y, z, S for which $y < z \leq \mathbf{V}_{s \in S} s$ there exists a finite subset $F \subseteq S$ such that $y \wedge \mathbf{V}_{f \in F} f < z \wedge \mathbf{V}_{f \in F} f$. The author obtains as a main result that the condition of Baer is equivalent to the condition that L is meet-continuous. Furthermore the author introduces a more general condition which contains Baer's condition as a special case. Let $\mathfrak{S}$ be a fixed family of subsets S of L. Then L satisfies $C_{\mathfrak{S}}$ if for each system $y, z, S, S \in \mathfrak{S}$, such that $y < z \leq \mathbf{V}_{s \in S} s$ there exists an element $s_0 \in S$ such that $y \wedge s_0 < z \wedge s_0$. Let $\mathfrak{F}$ be the family of all finite subsets of L, let $\mathfrak{G}$ be the family of all directed subsets of L and let $\mathfrak{P}$ be the family of all subsets of L. The following results are proved for a complete lattice L: (i) L satisfies $C_{\mathfrak{F}} \Leftrightarrow L$ is distributive; (ii) L satisfies $C_{\mathfrak{G}} \Leftrightarrow L$ is meet-continuous $\Leftrightarrow L$ satisfies the condition of Baer; (iii) L satisfies $C_{\mathfrak{P}} \Leftrightarrow L$ is pseudo-relatively complemented. The following known results are corollaries of (ii) and (iii). The complete lattice of subalgebras and the complete lattice of congruence relations of a universal algebra satisfy the condition of Baer. A complete lattice is pseudo-relatively complemented if and only if it is meet-continuous and distributive.

Ph. Dwinger (Lafayette, Ind.)

4119:

Grätzer, György. Standard ideals. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **9** (1959), 81–97. (Hungarian)

An element s of a lattice L is called standard if it satisfies one of the following equivalent conditions: (α)

$$x \cap (s \cup y) = (x \cap s) \cup (x \cap y)$$

for any elements $x, y \in L$; (β) if $u, t \in L$ satisfy $u \leqq s \cup t$, then $u = (u \cap s) \cup (u \cap t)$; ($\gamma$) the relation θ_s defined as "$x \equiv y$ (θ_s) if and only if $(x \cap y) \cup s_1 = x \cup y$ for some $s_1 \leqq s$" is a congruence relation; (δ) for any two elements $x, y \in L$, (i) $s \cup (x \cap y) = (s \cup x) \cap (s \cup y)$, and (ii) from $s \cup x = s \cup y$, $s \cap x = s \cap y$ follows $x = y$. The standard elements form a distributive sublattice of L and homomorphisms carry standard elements into standard elements. Every neutral element is standard, while the converse statement is true if one assumes that the lattice is (weakly) modular or relatively complemented. An ideal S of L is called standard if it is a standard element in the lattice $\mathscr{L}$ of all ideals of L. An equivalent definition is that, for any (principal) ideal I of L, the elements of $S \cup I$ are of the form $s \cup x$ ($s \in S$, $x \in I$). If S is a standard ideal, then the ideal lattice of L/S (=the factor lattice with respect to the minimal homomorphism with S as kernel) is isomorphic to the interval $[S, L]$ of $\mathscr{L}$. If L is a relatively complemented lattice with 0 and 1, then the lattice of its congruence relations is a Boolean algebra if and only if every standard ideal of L is principal. Finally, the author shows that the two isomorphism theorems, the Zassenhaus lemma and the Jordan-Hölder-Schreier refinement theorem hold for standard ideals.

L. Fuchs (Budapest)

4120:

Maeda, Shûichirô. On the lattice of projections of a Baer *-ring. J. Sci. Hiroshima Univ. Ser. A **22**, 75–88 (1958).

In an earlier paper [J. Sci. Hiroshima Univ. Ser. A **19** (1955), 211–237; MR **17**, 1177] the author showed that in any complete lattice with abstract relations of independence and equivalence there can be introduced a dimension function. It was shown there that both continuous geometries and the lattices of projections of AW^* algebras are of the type considered. The present paper considers the problem of giving conditions for Baer *-rings which will guarantee that the projection lattice will be of this type. The conditions which are given here are general enough to include the most important Baer *-rings. Further, a reduction for Baer *-rings is developed, which includes the theorem of Berberian concerning the regular ring of a finite AW^* algebra [Trans. Amer. Math. Soc. **83** (1956), 493–509; MR **19**, 47].

F. B. Wright (New Orleans, La.)

4121:

Varsavsky, Oscar. Quantifiers and equivalence relations. Rev. Mat. Cuyana **2** (1956), 29–51 (1958).

The author reviews the work of P. Halmos [Compositio Math. **12** (1956), 217–249; MR **17**, 1172] and C. Davis [Amer. J. Math. **76** (1954), 747–762; MR **16**, 324] on monadic algebras, Boolean algebras, etc. He studies in detail properties of open and closed equivalence relations and the existence of individuals in a monadic algebra. Several conditions are explored which are either necessary and sufficient or only sufficient (but more easily applied) for maximal free filters to be individual constants. Several useful examples are described.

B. A. Galler (Ann Arbor, Mich.)

CLASSICAL ALGEBRA

4122:

★Dickson, Leonard E. Algebraic theories. Dover Publications, Inc., New York, 1959. ix+276 pp. Paperbound: $1.50.

Unaltered republication of the 1st ed. (entitled *Modern algebraic theories*) [Sanborn, Chicago-New York, 1926].

THEORY OF NUMBERS

See also 4189, 4252, 4338, 4520.

4123:

★Carmichael, Robert D. The theory of numbers and Diophantine analysis. Dover Publications, Inc., New York, 1959. 94+vi+118 pp. Paperbound: $1.35.

Unaltered republication, bound as one volume, of *The theory of numbers* [1st ed., Wiley, New York, 1914] and *Diophantine analysis* [1st ed., Wiley, New York, 1915].

4124:

Sengenhorst, P. Bemerkung zu der Arbeit von Herrn G. Kirschmer "Über eine mit den Pythagoräischen Zahlen zusammenhängende Gruppe". Elem. Math. **13** (1958), 105–106.

Modification, using complex numbers, of Kirschmer's construction [Elem. Math. **12** (1957), 49–56; MR **19**, 12].

4125:

Sierpiński, W. On Pythagorean triangles with equal areas. Wiadom. Mat. (2) **1** (1955/56), 163–168. (Polish)

The author generalizes Fermat's theorem on incongruent equiareal Pythagorean triangles to: if among $n \geqq 1$ equiareal Pythagorean triangles with distinct hypothenuses at least one has a hypothenuse of odd length, then there exist $n+1$ equiareal Pythagorean triangles with distinct hypothenuses one of which has odd length. Some questions concerning primitive, isoperimetric, minimal, etc., sets of Pythagorean triangles are also considered and answered.

Z. A. Melzak (Murray Hill, N.J.)

4126:

Sierpiński, W. On the equation $3^x + 4^y = 5^z$. Wiadom. Mat. (2) **1** (1955/56), 194–195. (Polish)

Working modulo 2 the author proves that the only solution in positive integers is $x = y = z = 2$.

Z. A. Melzak (Murray Hill, N.J.)

4127:

Moser, Leo. Notes on number theory. I. On the product of the primes not exceeding n. Canad. Math. Bull. **2** (1959), 119–121.

Let $R(n)$ be the product of all primes $p \leqq n$. Then it is well known that $R(n) < c^n$ with a constant c. Only analysing the number $(6m+1)!/m!(2m)!(3m!)$ by very elementary tools the author shows that here c may be chosen such that $c^5 = 2^4 \cdot 3^3$ (i.e. $c < 3.37$). The best known result given by Rosser is $c = 2.83$.

H. Bergström (Göteborg)

4128:

Vandiver, H. S.; and Weaver, Milo W. Introduction to arithmetic factorization and congruences from the standpoint of abstract algebra. Amer. Math. Monthly **65** (1958), no. 8, part II, 53 pp.

Quite apart from its being fitting that the Slaught committee should select a manuscript on number theory to occupy a prime place in the sequence of Herbert Ellsworth Slaught Memorial Papers, it is about time that something be published of an elementary, expository nature in which the tools of group (or semigroup, or ring, etc.) theory are used to obtain theorems in elementary number theory. There are of course those who argue that group theory should be learned after having been motivated by a knowledge of number theory. A strict insistence on this point of view renders the present monograph useless, but the group-theoretic demands made on the reader are not excessive. One pays in this case a very small price to get interesting proofs.

There are fewer than 49 full pages of mathematics, and a fairly long beginning segment of this is concerned with fundamentals before one finds what could be called a real application of groups to numbers. Pages 12 through 47 form the heart of the classical matter, and in these we find (among other things) properties of congruences, Euler's φ-function, Wilson's theorem, the Minkowski-Thue theorem, Fermat's theorem on the representation of primes as a sum of squares, primitive roots, Carmichael numbers, and quadratic reciprocity.

One cannot help but be thankful for the expert manner in which the authors practice their craftsmanship, once they are in the swing of things. For those who know the elements of both algebra and number theory, this paper will make pleasant reading.

However, the shortcomings are many—though they center not so much about the mathematics as in the manner of presentation. The reviewer feels compelled to dwell upon such matters because of a personal feeling about what the general nature of the Slaught series ought to be: well-written expository monographs for the non-specialist. There are theorems stated in terms of concepts never really used, proofs which include irrelevances, and definitions or descriptions which need repair. These are minor. Worse, and not at all minor, the writing is too often elliptical (there are many instances, but as a shortest example, one complete sentence reads: Assume first that $m=p^n$, where p is prime and k_1 is an integer); and while this will tend only to plague the beginner, it seems inexcusable for the Slaught series especially.

There are problems, and most are good ones, but their distribution according to sections is, for purposes of instruction, too far from rectangular. The authors simply chose not to include several "natural" problems.

Chapters VIII and IX (the last two) constitute a rather disparate segment, but their content is of interest for those who like to generalize basic number-theoretic ideas.

In chapter VIII, it is shown that the nonunits in $Z/(n)$ form a multiplicative semigroup, and there is a theorem about its structure. Primes, units, and associates are then defined for certain semigroups in a natural way and a unique factorization theorem is proved for $Z/(n)$.

In chapter IX, there is presented a description of a semiring (semigroup relative to addition, semigroup relative to multiplication, and distributive law as usual) which, as a notion, is an extension of the ring $Z/(n)$, but nothing is done with it.

It seems too bad that the book does not end in a chapter on Gaussian integers, where concepts from rings and ideals can be utilized to obtain interesting results and where the power of certain theorems from the first few chapters could be seen in a different setting.

R. J. Wisner (Princeton, N.J.)

4129:

Hanneken, C. B. Irreducible sextic congruences. Duke Math. J. **26** (1959), 81–93.

In an earlier paper [same J. **22** (1955), 107–118; MR **16**, 674] the author has classified irreducible 5-ic congruences under the group of linear functional transformations with coefficients in a modular field defined by a prime p. Here he considers the same problem for 6-ic congruences (when $p \neq 2,3$). The number of irreducible sextics in a conjugate set is called the order of the conjugate set. The number of conjugate sets of a given order is determined. The total number of distinct conjugate sets is $(p^3+p^2+3p+1)/6$ if p is of the form $6k+1$ and $(p^3+p^2+3p-3)/6$ if p is of the form $6k-1$.

H. Bergström (Göteborg)

4130:

Hanneken, C. B. Irreducible congruences over GF(p). Proc. Amer. Math. Soc. **10** (1959), 18–26.

The problem considered in the paper of the author reviewed above is here treated generally for m-ic congruences and is studied more systematically.

H. Bergström (Göteborg)

4131:

Toscano, Letterio. Sulle soluzioni intere dell'equazione $4x^3=27y^2+N$. Riv. Mat. Univ. Parma **8** (1957), 405–406.

The author modifies the method of Bombieri [same Riv. **8** (1957), 199–206; MR **20** #6388] to give a simpler proof of the same result.

J. W. S. Cassels (Cambridge, England)

4132:

Chowla, P.; Chowla, S.; Dunton, M.; and Lewis, D. J. Some diophantine equations in quadratic number fields. Norske Vid. Selsk. Forh. **31** (1958), 181–183.

The authors are interested in the following general problem. For $n \geq 3$, let $S_n+aS_{n-1}+bS_{n-2}=0$, with a, b, S_1, S_2 arbitrary but fixed integers. Determine an explicit $f=f(a, b, c, S_1, S_2)$ such that the equation $S_n=c$ has no solutions for $n>f$. This note deals with the special case $a=-1$, $b=\lambda \geq 2$, $S_1=1$, $S_2=1-2\lambda$, $c=1$. For $\lambda>2$, $f=1$. For $\lambda=2$, $f=4$. This result was proved but not published by Artin and Fox, and the methods used were quite different.

N. J. Fine (Princeton, N.J.)

4133:

Chowla, Paromita. A class of Diophantine equations. Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 569–570.

Let $\alpha=\frac{1}{2}(1+\sqrt{7})$, $\beta=\frac{1}{2}(1-\sqrt{7})$ and $s(g)=\alpha^g+\beta^g$. The author outlines the proof of the result that for $n>2$ the only solution of $s(g)=s(2^n)$ is $g=2^n$.

She also can prove that $s(g)=c$ has at most two solutions, but does not give the proof.

P. Erdős (Budapest)

4134:

Kneser, Martin. Klassenzahlen quadratischer Formen. Jber. Deutsch. Math. Verein. **61** (1958), Abt. 1, 76–88.

This is an expository paper which, after outlining the fundamental concepts for forms with coefficients over an integral domain of characteristic not 2, deals with the reduction theory, the Minkowski-Siegel-Maass formula and spinor genera. Except for the last topic, the emphasis is on positive forms. *B. W. Jones* (Mayaguez, P.R.)

4135:

Ehlich, Hartmut. Ein elementarer Beweis des Primzahlsatzes für binäre quadratische Formen. J. Reine Angew. Math. **201** (1959), 1–36.

The author gives an elementary proof of the prime number theorem for quadratic forms, namely:

$$\pi(x, D, f) \sim \frac{x}{\varepsilon h(\log x)},$$

where $h = h(D)$ is the number of classes of equivalent primitive forms of discriminant D, $f = au^2 + buv + cv^2$, $D = b^2 - 4ac$, ε is 1 or 2 according as f is non-ambiguous or ambiguous, and $\pi(x, D, f)$ denotes the number of primes not greater than x which are represented by the form. The proof uses in no sense obvious analogues of some results of Selberg.

An immediate consequence of this result is Dirichlet's theorem that an infinite number of primes are represented by any primitive binary form, for which an elementary proof was given by W. E. Briggs [Canad. J. Math. **6** (1954), 353–363; MR **16**, 117]. The author points to an "error" in Briggs' paper, which seems really to be a misinterpretation of the following sums of Briggs:

$$\sum \omega_n = \sum \lambda_d \sum 1,$$

where the first sum is over all $n \leqq x$, $n = f$, $(n, 2D) = 1$, the third sum is over all divisors of n with the above restrictions on n and the second sum is over all $d \leqq x$ with $(d, 2D) = 1$; $\lambda_d = \mu(d) \log^2 (x/d)$ and ω_n is the sum of λ_d over the divisors, d, of n. The author draws unwarranted and incorrect conclusions from Briggs' results. But the author's proof of the prime number theorem seems to be valid. *B. W. Jones* (Mayaguez, P.R.)

4136:

Davenport, H. Cubic forms in thirty-two variables. Philos. Trans. Roy. Soc. London. Ser. A **251** (1959), 193–232.

D. J. Lewis [Mathematika, **4** (1957), 97–101; MR **20** #3827], B. J. Birch [ibid. 102–105; MR **20** #3828] and Davenport have independently proved the existence of a number N such that every cubic form in n variables, with integral coefficients, is a zero form if $n \geqq N$. Lewis' proof is based on a theorem of R. Brauer [Bull. Amer. Math. Soc. **51** (1945), 749–755; MR **7**, 108]; Birch proves the analogous result for all forms of odd degree with a value of N depending on the degree of the form. The present paper contains Davenport's proof, which leads to the value $N = 32$. His result is as follows. Let

$$C(\mathbf{x}) = C(x_1, \cdots, x_n) = \sum_{i,j,k=1}^{n} c_{ijk} x_i x_j x_k$$

be a cubic form in n variables with integral coefficients. Then if $n \geqq 32$ there exist integers $x_1, \cdots, x_n$ such that

$$C(x_1, \cdots, x_n) = 0.$$

It is known [Mordell, J. London Math. Soc. **12** (1937), 127–129] that the theorem is false for $n < 10$. Tartakowsky [Izv. Akad. Nauk SSSR **1935**, 483–524] had attempted to apply the Hardy-Littlewood method to "general" forms, but "his work appears to be incomplete". Most applications of the Hardy-Littlewood method are to additive problems; the difficulty now is that we are dealing with forms that contain product terms. In the next place, it is observed that the Hardy-Littlewood method, when it is effective, yields an asymptotic formula, the main term of which is the product of two factors. One of these essentially measures the density of the solutions in the real field. The other is an infinite product over the primes; each factor measures the density of the solutions in the p-adic field. For the present application it is necessary that the given equation have a non-singular solution both in the real field and in every p-adic field. It is observed that solvability in the p-adic field has been proved by Demyanov [Dokl. Akad. Nauk SSSR **74** (1950), 889–891; MR **12**, 315] and Lewis [Ann. of Math. (2) **56** (1952), 473–478; MR **14**, 251].

We now briefly sketch the proof of the theorem. § 1 of the paper is introductory. § 2 is devoted to the p-adic problem. Assume that $C(\mathbf{x})$ is not degenerate, that is, $C(\mathbf{x})$ is not equivalent to a cubic form in fewer than n variables; that is certainly the case if $C(\mathbf{x})$ does not represent zero. Let $N = \frac{1}{2}n(n+1)$ and let $\mathscr{C}$ denote the matrix of n rows and N columns (c_{ijk}), where i indicates the row and the pair j, k, with $j \leqq k$, indicates the column. Let $h(C)$ denote the greatest common divisor of all determinants of order n formed from $\mathscr{C}$; then it is proved that $h(C)$ is an invariant of C; that is, it has the same value for any two equivalent forms. Next if l is any positive integer, the form $C(\mathbf{x})$ is said to have the property $\mathscr{A}(p^l)$ if there exist integers $x_1, \cdots, x_n$ such that

$$C(x_1, \cdots, x_n) \equiv 0 \pmod{p^{2l-1}},$$

$$\frac{\partial C}{\partial x_1} \equiv \cdots \equiv \frac{\partial C}{\partial x_n} \equiv 0 \pmod{p^{l-1}},$$

$$\frac{\partial C}{\partial x_i} \not\equiv 0 \pmod{p^l} \text{ for some } i.$$

The main result of § 2 is that any non-degenerate cubic form in at least 10 variables has the property $\mathscr{A}(p^l)$ for every p and a suitable l depending on p. Moreover an upper bound for l is obtained that depends on $h(C)$ but not on p.

§ 3 is concerned with the general cubic exponential sum

$$S = \sum_{x_1, \cdots, x_n} e(\Gamma(x_1, \cdots, x_n)),$$

where $\Gamma(\mathbf{x})$ is a cubic form with real coefficients, $e(\alpha) = e^{2\pi i\alpha}$ and the summation is over all x_j such that $Px_j' \leqq x_j < Px_j''$. The aim is to investigate the consequences of the hypothesis $|S| \geqq P^{n-k}$, where k is a fixed positive number, small in comparison with n.

In § 4 the results of § 3 are applied to two types of exponential sums. The first type is the sum S with $\Gamma(\mathbf{x}) = \alpha C(\mathbf{x})$, where α is real. The second sum is

$$S_{a,q}(\mathbf{l}) = \sum_{\mathbf{x}} e\left\{\frac{a}{q} C(\mathbf{x}) + \frac{l_1 x_1 + \cdots + l_n x_n}{q}\right\},$$

where each x_j runs through a complete residue system (mod q). It is assumed throughout that $C(\mathbf{x})$ does not represent zero.

§ 5. Approximation on major arcs.

§ 6. The choice of intervals. One of the lemmas in this section asserts the existence of a non-singular real solution of the equation $C(\mathbf{x})=0$.

§ 7. The singular series. This is defined by

$$\mathfrak{S} = \sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} q^{-s} S_{a,q}, \quad S_{a,q} = S_{a,q}(\mathbf{0}).$$

For $n \geqq 10$ and $C(\mathbf{x})$ non-degenerate it is proved that $\mathfrak{S}>0$.

§ 8. The proof of the theorem is now accomplished in several stages. It is proved first that if $C(\mathbf{x})$ does not represent zero then it represents a form of the type

$$C'(x_1, \cdots, x_{n-3}) + d_1 y_1^3.$$

Applying the same process to $C'(\mathbf{x})$ we ultimately arrive at a form

$$d_1 y_1^3 + \cdots + d_3 y_3^3.$$

The final step is the proof that this form represents zero.

L. Carlitz (Durham, N.C.)

4137:

Rubinstein, Elchanan. On binary forms of degree n, $n \geqq 3$. Riveon Lematematika **12** (1958), 41–72. (Hebrew. English summary)

Let (I) $F(x, y)=m$ be a representation of the integer m by a binary form of degree n. By considering transformations of the form: (II) $x=m_1^{\nu}u+ry$, where $m_1|m$ and $0 \leqq r < m_1^{\nu}$, ν an arbitrary integer, the author obtains (as for quadratic forms) a classification of all solutions of (I) depending on the solutions of the congruence: (III) $F(r, 1) \equiv 0 \pmod{m_1}$. Two solutions, (x, y), (x_1, y_1) belong to the same class (are homologous) if and only if $xy_1 - yx_1 \equiv 0 \pmod{m_1^{\nu}}$, and each class determines a unique solution (mod m_1^{ν}) of (III). Elementary bounds are given for the numbers of homologous classes of solutions. Furthermore, it is shown that if $m=m_1^{\nu_1}m_2$, where $\nu > \nu_1 \geqq 1$ and $(m_1, m_2)=1$, can be represented by $F(x, y)$ only a finite number of times, then applying transformations (II) one obtains a finite number of binary forms of the same degree and of discriminant $D'=m_1^{(n-1)(n-2\nu_1)}D$ (D = discriminant of $F(x, y)$) which represent m_2 only once or twice. The latter happens only if the degree of F is even, and the two representations of m_2 are given by (x, y) and $(-x, -y)$.

S. A. Amitsur (New Haven, Conn.)

4138:

Amitsur, S. A. On arithmetic functions. J. Analyse Math. **5** (1956/57), 273–314.

Verf. verallgemeinert den Selbergschen Beweis des Primzahlsatzes auf eine gewisse Klasse von arithmetischen Funktionen. Er benutzt die Beweisanordnungen von H. N. Shapiro [Ann. of Math. (2) **51** (1950), 485–497; **52** (1950), 217–230, 231–243; MR **11**, 419; **12**, 81] und K. Yamamoto [J. Math. Soc. Japan. **7** (1955), 424–434; MR **18**, 18]. Es wird dabei einer arithmetischen Funktion $g(n)$ eine durch

$$(S_g f)(x) = \sum_{n \leqq x} g(n) f\left(\frac{x}{n}\right)$$

definierte lineare Transformation im linearen Raum $\mathfrak{R}$ der für $x \geqq 1$ definierten und auf jedem endlichen Intervall beschränkten komplexwertigen Funktionen f zugeordnet. Der Faltung ("convolution")

$$(g * h)(x) = \sum_{d|n} g(d)\, h\left(\frac{n}{d}\right)$$

zweier arithmetischer Funktionen g und h entspricht das Produkt der zu g und h gehörigen linearen Transformationen. Die Algebra der arithmetischen Funktionen (mit "Faltung" statt "gewöhnlicher Multiplikation") ist isomorph mit der Algebra A der zugehörigen linearen Transformationen von $\mathfrak{R}$. Die Multiplikation einer arithmetischen Funktion mit $\log x$ (oder allgemeiner mit einer additiven Funktion) induziert in A eine Ableitung ("derivation"). Verf. betrachtet jetzt den Teilraum $\mathfrak{L}$ von $\mathfrak{R}$, der aus allen Polynomen in $\log x$ besteht. Mit Hilfe eines symbolischen Kalküls erhält er (für die in Betracht kommenden arithmetischen Funktionen g) asymptotische Formeln für $I_g f$ (wo $I_g = S_{gs^{-1}}$), in denen die Hauptglieder Funktionen aus $\mathfrak{L}$ sind. Diese Formeln entsprechen der Selbergschen fundamentalen Ungleichung. Der Hauptsatz des Verf. verkörpert eine Verallgemeinerung des Verfahrens, mit dem Selberg den Primzahlsatz aus der fundamentalen Ungleichung herleitet. Er lautet folgendermaßen: Es sei f eine beschränkte Funktion aus $\mathfrak{R}$ mit den folgenden Eigenschaften:

$$(1) \qquad |f(x)| \log x \leqq \sum_{n \leqq x} \Lambda(n) f\left(\frac{x}{n}\right) + o(\log x)$$

(Λ ist die aus der Primzahltheorie bekannte Faltung der Funktion μ von Moebius mit $\log x$); (2) Es gibt ein positives ganzes k derart, daß

$$\sum_{n \leqq x} \left| f\left(\frac{x}{n}\right) - f\left(\frac{x}{n+1}\right) \right| = O(\log^k x);$$

(3) $\sum_{n \leqq x} f(n)/n = O(1)$; (4) $|f(tx)-f(x)| = o(1)$ falls $t \to 1$ und $x \to \infty$. Dann gilt $f(x)=o(1)$. Falls $f(x) = \sum_{n \leqq x} \mu(n)/n$ erhält man den Primzahlsatz. Verf. beweist auch den Dirichletschen Satz über Primzahlen in einer arithmetischen Progression.

H. D. Kloosterman (Zbl **81**, 43)

4139:

Knapowski, S. On new "explicit formulas" in prime number theory. I. Acta Arith. **5** (1958), 1–14 (1959).

Let $\Lambda(n)=\log p$ if $n=p^m$, where p is a prime, $m=1, 2, \cdots$, and let $\Lambda(n)=0$ otherwise. For $x>1$, let $\psi(x) = \sum_{n \leqq x} \Lambda(n)$. Let $\psi_0(x) = \frac{1}{2}[\psi(x-0)+\psi(x+0)]$, and $U_N(s) = \sum_{n \leqq N} n^{-s}$. The author proves the following result. For $2 \leqq x \leqq N$, $N \geqq N_0$, we have

$$\psi_0(x) = N^{-1}(\log N!) - \sum_{\rho} (x^{\rho}/\rho),$$

with the understanding that $\rho=\beta+i\gamma$ stands for the zeros of $U_N(s)$, and $\sum_{\rho}(x^{\rho}/\rho)$ stands for $\lim_{T \to \infty} \sum_{|\gamma| \leqq T} (x^{\rho}/\rho)$. The use of the zeros of the "Dirichlet polynomial" $U_N(s)$ instead of the zeros of the zeta-function is suggested by the work of P. Turán [*Eine neue Methode in der Analysis und deren Anwendungen*, Akadémiai Kiadó, Budapest, 1953; MR **15**, 688]. This result implies a new approximation for $\psi_0(x)-x$.

K. Chandrasekharan (Bombay)

4140:

Pan, Cheng-tung. On the least prime in an arithmetical progression. Sci. Record (N.S.) **1** (1957), 311–313.

Let $P_{\min}(D, l)$ denote the least prime in an arithmetic progression $nD+l$, where $1 \leq l \leq D-1$ and $(l, D)=1$. K. A. Rodosskiĭ showed in 1954 that there exists a positive absolute constant A such that $P_{\min}(D, l) < D^A$ but gave no indication of the size of A. In the paper here reviewed the author announces the result that $A \leq 10^4$. A detailed proof is not given but the necessary fundamental lemmas needed to obtain the result are mentioned without proof.

W. H. Simons (Vancouver, B.C.)

4141:

Luthra, S. M. Partitions of bipartite numbers when the summands are unequal. Proc. Nat. Inst. Sci. India. Part A **23** (1957), 370–376.

Let $p(m, n)$ be the number of unrestricted partitions of the bipartite number (m, n), and let $q(m, n)$ be the number of partitions of (m, n) into distinct parts. F. C. Auluck found asymptotic expressions for $p(m, n)$ in the two cases: (a) m fixed; (b) m and n of the same order [Proc. Cambridge Philos. Soc. **49** (1953), 72–83; MR **14**, 726], and the author, using Auluck's methods, now obtains asymptotic expressions for $q(m, n)$ in the same two cases.

H. Halberstam (London)

4142:

Schiller, James K. A theorem in the decimal representation of rationals. Amer. Math. Monthly **66** (1959), 797–798.

The author considers the number of times a given digit appears in the repeating sets that occur in the decimal representation of all rational numbers of the form m/p where p is a prime, $(10m, p)=1$, $m<p$. He sets $p=10q+r$, $0<r<10$, and proves that $11-r$ digits appear q times and the remaining $r-1$ digits appear $q+1$ times. For example, if $r=3$ the digits 3 and 6 will each appear $q+1$ times and the other digits will each appear q times.

H. W. Brinkmann (Swarthmore, Pa.)

4143:

Hasse, Helmut. Der 2^n-te Potenzcharakter von 2 im Körper der 2^n-ten Einheitswurzeln. Rend. Circ. Mat. Palermo (2) **7** (1958), 185–244.

The problem of giving a criterion for the 2^n-th power residue character of the integer 2 modulo a prime $p \equiv 1 \pmod{2^n}$ in the field P of rational numbers has until this day been solved only in the special cases $n=1, 2, 3, 4$. In the formulation of such criteria certain quadratic partitions of p are utilized. For instance, a result of Aigner [Deutsche Math. **4** (1939), 44–52] states that 2 is a 2^4-th power residue of a prime $p \equiv 1 \pmod{2^4}$ if and only if p is simultaneously representable in the forms

$$p = (1-2^2x)^2+(2^5y)^2 = (1-2^2u)^2+2(2^3v)^2$$

or in the forms

$$p = (1-2^2x)^2+(2^4\bar{y})^2 = (1-2^2u)^2+2(2^2\bar{v})^2,$$

where $\bar{y}$, $\bar{v}$ are odd integers.

In the present paper there is developed an extensive and penetrating study of the 2^n-th power residue symbol $(2/\alpha)_{2^n}$ with respect to a number α prime to 2 in the algebraic number field P_{2^n} generated by a primitive 2^n-th root of unity. Relative norm and semi-norm formations over P_{2^n} are employed to describe a procedure for decomposing α into coordinates in real subfields of P_{2^n}. These coordinates are analogous to the numbers y and v in the equations above. Use is made of "Grössencharaktere" constructed from Jacobi sums in P_{2^n} in accordance with a scheme of A. Weil [Trans. Amer. Math. Soc. **73** (1952), 487–495; MR **14**, 452]. Combining the remarkable Jacobi sum identities of Davenport and Hasse [J. Reine Angew. Math. **172** (1934), 151–182] with the sharp formula of Artin and Hasse [Abh. Math. Sem. Univ. Hamburg **6** (1928), 146–162] for the second supplementary theorem of the law of higher reciprocity for prime powers, the author derives an explicit representation for $(2/\alpha)_{2^n}$ in terms of the coordinates of α. By specializing, he then deduces the known criteria in the rational cases corresponding to $n=1, 2, 3, 4$. For large values of n the computational difficulties are increasingly formidable. The author makes a detailed analysis of the case $n=5$, but he does not completely settle the question of obtaining an analogous criterion in this case.

{Reviewer's note: To the author's introduction should be added the statement that Reuschle's criterion for the octic character of 2 was first proved by J. Perott [J. Reine Angew. Math. **102** (1888), 185–223].}

A. L. Whiteman (Princeton, N.J.)

4144:

Legendre, Robert. Fonctions fuchsiennes de la deuxième famille. C. R. Acad. Sci. Paris **248** (1959), 3097–3098.

"Une précédente Note [mêmes C. R. **247** (1958), 770–772; MR **20** #5279] a montré que les fonctions fuchsiennes symétriques de la deuxième famille s'expriment à l'aide de fonctions thêtafuchsiennes simples. Ce résultat est généralisable à toutes les fonctions de la deuxième famille." (Author's summary)

H. Cohn (Tucson, Ariz.)

4145:

Chalk, J. H. H.; and Erdös, P. On the distribution of primitive lattice points in the plane. Canad. Math. Bull. **2** (1959), 91–96.

Let θ be a given irrational number and α be any real number; then it is well known that there exists an absolute constant λ such that

$$x|y-\theta x+\alpha| < \lambda$$

is satisfied by infinitely many integers x, y with $x>0$ [for the latest results on this classical problem see J. W. S. Cassels, Math. Ann. **127** (1954), 288–304; MR **15**, 687]. The authors show, under the same conditions for θ and α, that there exists an absolute constant λ such that the inequality

$$x|y-\theta x+\alpha| < \lambda (\log x/\log\log x)^2$$

is satisfied by infinitely many coprime integers x, y with $x>0$. The proof uses the continued fraction expansion of θ.

J. Popken (Amsterdam)

4146:

Godwin, H. J. The determination of fields of small discriminant with a given subfield. Math. Scand. **6** (1958), 40–46.

The author extends the methods developed in his earlier papers [e.g., Quart. J. Math. Oxford Ser. (2) **8** (1957), 214–222; MR **20** #3844] to obtain bounds generally for the roots, and hence for the coefficients, of polynomials which determine fields of a given type, having a prescribed subfield. As an application, he proves that a totally real sextic field which contains the quadratic field generated by $5^{1/2}$ either has discriminant 300125, or has discriminant not less than 355556, and that the only totally real sextic field of discriminant 300125 is generated by $5^{1/2}$ and $\cos 2\pi/7$. The author conjectures that 300125 is the minimum of discriminants of totally real sextic fields. *R. Hull* (Santa Monica, Calif.)

4147:

Schmidt, Hermann. Über spezielle mehrgradige Gleichungen und Quadratsummendarstellungen in imaginär quadratischen Körpern. Math. Nachr. **19** (1958), 323–330.

Let $L \geqq 1$, $Q \geqq 1$ be rational squarefree integers. It is proved that $-L$ can be represented as the sum of two squares of numbers in the imaginary quadratic field $P(\sqrt{-Q})$, except when (1) there exists a prime $p \equiv 3 \pmod 4$ such that $p|L$, $p \nmid Q$, $\left(\frac{Q}{p}\right) = -1$, or (2) L has an even number of prime divisors $\equiv 3 \pmod 4$. Two proofs are given. One depends on an elementary reduction of the problem to the non-trivial solvability of a certain diophantine equation

$$a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2 = 0,$$

for which classical results are available [e.g. L. E. Dickson, *Studies in the theory of numbers*, Univ. of Chicago Press, 1930, theorem 61; T. Skolem, *Diophantische Gleichungen*, Springer, Berlin, 1938, Ch. 3, Theorem 3; B. W. Jones, *The arithmetic theory of quadratic forms*, Math. Assoc. Amer., Buffalo, N.Y., 1950; MR **12**, 244; Corollary 27c]. The second proof depends on p-adic methods [see K. Hensel, *Zahlentheorie*, Göschen, Berlin-Leipzig, 1913; H. Hasse, *Vorlesungen über Zahlentheorie*, Springer, Berlin-Göttingen-Heidelberg, 1949; MR **14**, 534; § 26; J. Reine Angew. Math. **153** (1924), 113–130; theorem 5]. The note concludes with some examples. [See also I. Niven, Trans. Amer. Math. Soc. **48** (1940), 405–417; MR **2**, 147; and O. Hemer, Ark. Mat. **2** (1952), 57–82; MR **14**, 247]. *H. Halberstam* (London)

FIELDS

See also 4122, 4146, 4182, 4319.

4148:

Rédei, L.; und Turán, P. Zur Theorie der algebraischen Gleichungen über endlichen Körpern. Acta Arith. **5** (1959), 223–225.

Rédei [*Algebra*, Vol. I, Akadémiai Kiadó, Budapest, 1954; MR **16**, 559] has proved that the number of distinct solutions in GF(q) of the equation

$$(*) \qquad a_0 + a_1x + \cdots + a_{q-2}x^{q-2} = 0 \quad (a_j \in \mathrm{GF}(q),\ a_0 \neq 0)$$

is equal to $q-1-r$, where r is the rank of the cyclic matrix

$$Z = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{q-2} \\ a_{q-2} & a_0 & a_1 & \cdots & a_{q-3} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ a_1 & a_2 & a_3 & \cdots & a_0 \end{bmatrix}.$$

The special case $q=p$ is the theorem of König and Rados [J. Reine Angew. Math. **99** (1886), 256–260]. In the present paper it is proved very simply that the number of distinct solutions of (*) is equal to $q-1-r_{\mathrm{hms}}$, where r_{hms} is the "Hauptminorsummenrang" of Z, that is, the maximum integer k for which the sum of the principal minors of Z of order k is different from zero. It follows that $r = r_{\mathrm{hms}}$. *L. Carlitz* (Durham, N.C.)

4149:

Kertész, A. Simple proof of a fundamental theorem of field theory. Amer. Math. Monthly **66** (1959), 804.

Let L be an extension of a field K and let A and B be two subsets of L. Then A is algebraically dependent on B (over K) if the elements of A are algebraic over $K(B)$. If B is also algebraically dependent on A, then A and B are equivalent over K. The set A is algebraically independent over K if no element x of A is algebraically dependent on the set $A-(x)$. The author proves the following theorem due to Steinitz: If A and B are two subsets of L, algebraically independent and equivalent over K, then A and B have the same cardinal number. (Only the infinite case is of interest here.) The proof is based on the Exchange Theorem and the fact that $\aleph_0 n = n$ for any infinite cardinal n. *W. E. Jenner* (Lewisburg, Pa.)

4150:

Manin, Yu. I. Moduli of a field of algebraic functions. Dokl. Akad. Nauk SSSR **125** (1959), 488–491. (Russian)

Soit R un corps de fonctions algébriques à une variable, défini sur un corps K, extension finie du corps des nombres rationnels. En supposant que R n'est pas hyperelliptique, l'auteur démontre l'existence d'un corps de définition pour R de dimension (sur les rationnels) $\leqq 3g-3$, où g est le genre de R ($g>1$). De plus, on obtient un système de paramètres (modules) spécial. Comme application on démontre une hypothèse de Teichmüller dans le cas non-hyperelliptique [voir J. Reine Angew. Math. **185** (1943), 1–11; MR **5**, 37; hypothèse no. 2] (dans le cas hyperelliptique cette hypothèse a été demontrée par M. Eichler [ibid., 12–13; MR **5**, 38]). *E. Lluis* (México, D.F.)

4151:

Nagata, Masayoshi. On the 14-th problem of Hilbert. Amer. J. Math. **81** (1959), 766–772.

Let s be an integer $\geqq 4$; set $r = s^2$, and let $P_1, \cdots, P_r$ be independent generic points of the projective plane S over the prime field (arbitrary characteristic). Then, if a curve C of degree d in S passes through each P_i with multiplicity at least $m > 0$, then $d > ms$. If $\mathfrak{p}_i$ is the homogeneous prime ideal of P_i in $k[x, y, z]$, and if $\mathfrak{a}_m = \bigcap_{i=1}^{r} \mathfrak{p}_i^m$, then there exists n such that $\mathfrak{a}_{ns} \neq \mathfrak{a}_m{}^n$. Using these facts, the author defines a group G of linear transformations of the polynomial ring $k[X_1, \cdots, X_r, Y_1, \cdots, Y_r]$ (k: field containing the coordinates of the P_i's), and proves that the ring $\mathfrak{o}$ of invariants of G is not finitely generated over

k. Thus Hilbert's Fourteenth Problem, in its original formulation, admits a negative answer. The quotient field of v has transcendence degree 4 over k. The author includes a history of Hilbert's 14th Problem, and conjectures that any integer $r > 9$ gives an analogous counter example.

P. Samuel (Urbana, Ill.)

4152:

Arf, Cahit. Über ein Analogon des Riemann-Rochschen Satzes in Zahlkörpern. Akad. Wiss. Mainz. Abh. Math.-Nat. Kl. **1957**, 293–328.

Let Q be the field of rational numbers. The author considers a topological group A which is essentially the same as the additive group of so-called valuation vectors (adèles) of Q. A pairing of A with itself is defined, and the annihilator of a subgroup B of A is denoted by $\hat{B}$. A closed subgroup F of A is then called almost isotropic if and only if both $F/F \cap \hat{F}$ and $\hat{F}/F \cap \hat{F}$ are finite groups. For such an F, the author proves the following analogue of the theorem of Riemann-Roch:

$$\dim F = \deg F + \dim \hat{F} + \log g.$$

Here g is a constant independent of F, and $\dim F$ [$\dim \hat{F}$] and $\deg F$ are the numbers attached to F [$\hat{F}$], defined similarly to the dimension and the degree of a divisor of an algebraic function field, respectively. A generalization to an arbitrary finite algebraic number field is also stated.

K. Iwasawa (Cambridge, Mass.)

ALGEBRAIC GEOMETRY

See also 4151, 4875.

4153:

Andreatta, Antonio. Le superficie cubiche singolari reali. Period. Math. (4) **37** (1959), 86–100.

Elementary study of real singular cubic surfaces.

M. Piazzolla-Beloch (Ferrara)

4154:

Bishara, S. On a certain quintic scroll. Proc. Math. Phys. Soc. Egypt. no. **21** (1957), 25–32 (1958). (Arabic summary)

L'Autore studia una particolare superficie rigata algebrica del quint'ordine Γ, dotata di una cubica doppia δ e di una retta tripla r, corda di δ. La direttrice tripla r è una generatrice doppia: per ogni suo punto generico passa una sola generatrice di Γ distinta da r, mentre ogni piano generico passante per r contiene due generatrici di Γ distinte da r. Con scelta opportuna del sistema di riferimento l'equazione di Γ può ridursi alla forma: $M_1\alpha^2 + M_2\alpha\beta + M_3\beta^2 = 0$ ove $\alpha \equiv zx - y^2$, $\beta \equiv yt - z^2$, $M_i \equiv m_i y + n_i z$.

D. Gallarati (Genova)

4155:

Kähler, Erich. Geometria aritmetica. Ann. Mat. Pura Appl. (4) **45** (1958). ix + 399 pp.

This, in more ways than one, is an unusual piece of work. By its size, it is a book; it appears as a volume in a journal. The author is German; the book appears in Italian. The subject combines algebra and geometry, with some arithmetical flavoring; but the author, instead of following in his terminology the accepted usage in either one of those subjects, or adapting it to his purposes, has chosen to borrow his vocabulary from philosophy, so that rings, homomorphisms, factor-rings, ideals, complete local rings appear as "objects", "perceptions", "subjects", "perspectives", "individualities". The book includes altogether new material along with much which turns out to be quite familiar (sometimes to the point of triteness) once it is translated back into more familiar language; but no attempt is made to point out what may be novel and what is not so; there are no historical or bibliographical indications, no "Leitfaden", no introduction apart from a two-page philosophical discourse which ends up with the following statement: "··· bibliographical references would probably have obscured the fact that a single philosophical tendency has been the real motive power behind the chain of my reasonings."

It seems clear that the book contains some grain and much chaff; to sift one from the other is a difficult task, which will have to be left largely to future readers, if the book finds any, as is to be wished; for it would be unfortunate if valuable material were to be lost to contemporary algebraic geometry because of the mode of presentation adopted by the author. An attempt will now be made to give an outline of the contents, chapter by chapter.

Chapters I and II give a brief elementary introduction to the theory of fields and of differentiation in fields and rings, including Fitting's definition of the differents of various orders [cf. H. Fitting, J-B. Deutsch. Math. Verein. **46** (1936), 195–228]. Chapter III explains the author's language concerning local rings ("objects", "subjects", "perspectives", etc.). Chapter IV introduces "varieties"; this is a concept which, for generality, stands roughly in between Weil's abstract varieties and Grothendieck's schemes, but with some features of its own. To define it, write $P(A)$, when A is a commutative ring with unit-element, for the set of the prime ideals in A which contain all the zero-divisors in A; for $p \in P(A)$, write A_p for the local ring consisting of all fractions a/b, where $a \in A$, $b \in A - p$; this is a subring of the ring of fractions of A (consisting of all fractions a/b, with $a \in A$, $b \in A$, b not a zero-divisor in A). Write $V(A)$ for the set of all local rings A_p with $p \in P(A)$. Now a "variety" V is a set of local rings (or "aspects") R, all having the same ring of fractions $\Omega(V)$ (the "object" of V), such that (a) if $R \in V$, then $V(R) \subset V$ and (b) if R, R' are in V, there is a subring A of $R \cap R'$ such that R and R' are in $V(A)$. A "variety" is said to be "algebraic" over a ring A if it is the union of finitely many "varieties" $V(A_i)$, where the rings A_i are finitely generated extensions of A with the same ring of fractions; V is said to be "arithmetic" if it is algebraic over the prime ring (the ring generated by the unit-element).

"Closed" varieties (a concept corresponding to the usual one of completeness) are defined as follows. A local ring R, with the maximal ideal P, is said to extend another one R' if $R \supset R'$ and if $P \cap R'$ is the maximal ideal in R'. A variety V is said to be "closed" [resp. "closed over a ring A"] if $\Omega(V)$ is a primary ring and if, to every local ring $R \subset \Omega(V)$ [resp. to every local ring $R \subset \Omega(V)$ containing A] there is a local ring $R' \in V$ and a local ring $R'' \subset \Omega(V)$ such that R'' extends both R and R'.

By a "point" of a variety V, the author understands what would more commonly be called a "complex-valued

place", i.e. an isomorphism into the complex number-field of the residue-field R/P of a local ring $R \in V$ with respect to its maximal prime ideal P. It is shown that, with a suitable topology, the set of "points" of V is a Hausdorff space, and a compact one if V is "closed" (a result which, even for abstract varieties, is nowhere to be found in print). There does not seem to be anything noteworthy in further developments concerning "analytic" varieties, which seem identical with varieties, in the usual sense, defined over the complex-number field, and which, naturally enough, are studied by the method of convergent power-series.

Chapter VI begins by introducing "figures": a "figure" on a variety V is a mapping which, to every local ring $R \in V$, assigns an ideal $I(R)$ in such a way that, if $R' \in V(R)$, then $I(R') = R' \cdot I(R)$. The remainder of the chapter is occupied by the elementary theory of ideals in local rings and seems to offer little, if anything, that is not well-known.

Chapter VII introduces the following concept (corresponding to that of proper morphism in modern algebraic geometry). Let V, W be two varieties, such that $\Omega(V)$, $\Omega(W)$ are primary rings and $\Omega(V) \supset \Omega(W)$; assume that, to every $R \in V$, there is $S \in W$ (which is then necessarily unique) such that R extends S, and write $S = f(R)$; assume also that, whenever a local ring $R \subset \Omega(V)$ extends a ring $S \in W$, there is a local ring $R' \in V$ and a local ring $R'' \subset \Omega(V)$ such that R'' extends both R and R'. Then the author says that V "extends" W and that f is a "projection" from V to W; the "extension" is called "algebraic" if, for every $R \in V$, there is a finitely generated extension A of $S = f(R)$ such that $R \in V(A)$. This concept is not exploited further, and the author proceeds to a series of lemmas concerning the lengths of variously defined modules in local rings, and the P-adic completion (the "individuality") of a local ring R with maximal ideal P. One goes back briefly to the global point of view, in order to introduce differents as "figures": if V is an "algebraic extension" of W, the different of order n of $R \in V$ over its projection $S = f(R)$ on W is a "figure" on V, written $D_n(V/W)$; similarly, for an "arithmetic" variety V, the different $D_n(V)$ is the figure attaching, to each $R \in V$, its different of order n over the prime ring. This concept is abandoned at once in favor of a fairly detailed consideration of the differents in local rings, which might well prove to be of interest to specialists of such questions. Then come more lemmas on lengths of various modules (some of which receive the name of "multiplicities of figures"), and in particular on Hilbert's characteristic function.

The brief Chapter VIII defines the "divisors" of a field K (not of a variety) over a ring A as the mappings from the set of all discrete valuations of K, trivial on A, to the integers. On a "closed" variety, every "figure" determines a "divisor"; in another digression, "finite" figures are defined (roughly speaking, this seems to mean the coherence of the corresponding sheaf); and "finite divisors" are defined as those which can be determined by a "finite" figure.

"The major problem of arithmetic geometry is to describe the structure of a field, i.e. to take cognizance of its essence, starting from the multiplicity of its aspects." Such is the opening statement of Chapter IX, which proves to be, nevertheless, the most interesting one of the book; it is chiefly devoted to the study of the author's generalization for the "differentials of the first kind" in classical geometry. If K is a field, a differential form attached to K, i.e. a sum of terms of the form $x_0 dx_1 \cdots dx_m$, with $x_0, x_1, \cdots, x_m$ in K, subject to the usual rules, is said to be "integral" if, for every discrete valuation-ring R in K, it can be similarly written in such a way that all the elements appearing in the new expression are in R. The additive group of integral differential forms of degree m is denoted by $D_m(K)$. If, in addition to the usual rules for differentials, one introduces the relations $dx = 0$ for all elements x of a subfield k of K, then one gets in a similar manner the integral differential forms of K over k, which make up, for each value m of the degree, a vector-space $D_m(K/k)$ over k. The differential forms can be identified in the usual manner with the antisymmetric tensors; definitions are also given for tensors of a more general kind, leading up to groups [resp. spaces] of integral tensors, denoted by $I_m(K)$ [resp. $I_m(K/k)$]. Some substantial results are given concerning the determination of $D_m(K)$, $I_m(K)$ [resp. $D_m(K/k)$, $I_m(K/k)$], when K is finitely generated over the prime field [resp. over k]. At first, the method is purely field-theoretic, K being represented as an algebraic extension of a purely transcendental extension of the groundfield; earlier results on the differents play an essential role here; it is proved in particular that, if K is finitely generated over the rational number-field, $D_m(K)$ and $I_m(K)$ are finitely generated groups. More precise results are obtained when K is the function-field attached to a complete non-singular variety in the usual sense, and culminate in the following theorems: if K is the function-field of a complete non-singular variety V, defined over an algebraic number-field k, then, for every m, $D_m(K/k)$ coincides with the space of differential forms of the first kind defined over k on V, and it is the vector-space generated over k by $D_m(K)$, which is itself, in virtue of the earlier results, a finitely generated module over the ring of integers of k. Similar results hold for $I_m(K/k)$ and $I_m(K)$. Not the least interesting part of this chapter is the detailed treatment of two examples, viz., the curve $x^m + y^m = 1$ over the field Q of rational numbers, and the elliptic curve $y^2 = x^3 + \theta$, with $\theta = (-23)^{1/2}$, over $k = Q(\theta)$; in both cases, the module $D_1(K)$ is completely determined; in the latter case, it is generated by $3\theta dx/y$. The chapter ends up with some consideration of the relations between differential forms and divisors, and between the differentials of a field and those of a subfield.

Chapter X discusses, in somewhat desultory fashion, birational correspondences, periods of integrals of the first kind in the classical sense, Siegel's modular functions, and the zeta-functions which have been introduced in recent years into the theory of varieties over finite fields and algebraic number-fields. The chief motivation here seems to be the author's feeling that he owes it to himself to have something to say on all these subjects. Whether a search for rough diamonds, among the pebbles with which this chapter is strewn, would prove profitable, is a question which the reviewer will not attempt to settle.

Such is this book: exasperating to read, exasperating to review. The language is strange, motivation is poor, the exposition is disorderly. Preliminary material, which should have been dealt with at an early stage, keeps cropping up at every turn of the road. There is no knowing whether a given passage is of substantial value or is an exercise in newspeak, whether it will be needed later

or is just thrown in for good measure. Two or three research papers, written in a more conventional style, could have covered the same ground much more adequately and more usefully. Fortunately there is a good index, and all concepts are quite sharply defined. Except for that, the author seems to have done everything in his power to discourage prospective readers, and is only too likely to have succeeded. The reviewer wishes again to emphasize that, while this book can in no way be recommended to the majority of mathematical readers, the few who will undertake its study with a definite purpose in mind may well find something in it to repay their trouble.

A. Weil (Princeton, N.J.)

4156:

Sampson, J. H.; and Washnitzer, G. Numerical equivalence and the zeta-function of a variety. Amer. J. Math. **81** (1959), 735–748.

The authors consider a non-singular projective variety V of dimension r, defined over a finite field k. It is shown that, if the group of r-cycles on $V \times V$, modulo numerical equivalence, is finitely generated (or even if a suitable subgroup of that group is finitely generated) the zeta-function of V is a rational function and satisfies a functional equation. On the other hand, it is shown that, if the logarithmic derivative of the zeta-function is a rational function, it has only simple poles. The paper includes a summary in cohomological language.

A. Weil (Princeton, N.J.)

4157:

Nishimura, Toshio. Note on a proof of a theorem of the Grassmann variety. Bull. Kyoto Gakugei Univ. Ser. B, no. 5 (1954), 22–23.

Der Verfasser macht eine kurze Bemerkung zu einem von Igusa im Jahre 1953 in einer Vorlesung gebrachten Beweis des Basissatzes für Graßmannsche Mannigfaltigkeiten. Der Referent vermißt einen Hinweis darauf, daß dieser Satz schon im Jahre 1941 von G. Aprile und Hodge bewiesen worden ist [s. F. Severi, *Geometria dei sistemi algebrici*, vol. II, Edizioni Cremonese, Rome, 1957; MR **21** #3409; cap. V].

W. Burau (Hamburg)

4158:

★Abhyankar, Shreeram. Ramification theoretic methods in algebraic geometry. Annals of Mathematics Studies, no. 43. Princeton University Press, Princeton, N.J., 1959. ix+96 pp. $2.75.

This monograph was developed from the notes for a one-semester course given by its author at Columbia University. It is designed for students with only a modest background in commutative algebra, but it undertakes to carry the reader through a proof of the uniformization theorem for algebraic surfaces over ground fields of characteristic zero via a direct route in which there are but a few well marked gaps. The long first chapter on ramification theory begins with a study of integral dependence and ends with the general theory of discriminants and the Galois theory of local rings. The second chapter on valuation theory is based largely on the work of Krull; it contains a treatment of simply ordered abelian groups, existence theorems, specialization and composition of valuations, ramification theory of valuations, and valuations of algebraic function fields. Chapter 3 begins with a sketch of dimension theory in noetherian local rings. Here reference is made to Northcott's *Ideal theory* [Cambridge Univ. Press, 1953; MR **15**, 390] for many of the proofs. The chapter closes with an introduction to the notion of quadratic transformation which is treated more elaborately in chapter 4, where Zariski's theorems on the limits of ascending sequences of normal local domains are discussed, and where, finally, a proof of the uniformization theorem is presented. This theorem asserts that if K is a two-dimensional algebraic function field over an algebraically closed ground field k of characteristic zero, and if v is a zero-dimensional valuation of K/k, then v can be uniformized; that is, a regular algebraic local domain (R, M) of dimension two can be constructed in such a way that v has center M in R, and K is the quotient field of R. The proof is an adaptation of the author's proof of this theorem in the more general case where k has arbitrary characteristic. After selecting an appropriate transcendence base (x, y) for K/k, a Galois extension K^* of $K_1 = k(x, y)$ such that $K^* \supseteq K$ is introduced. The restriction v_1 of v to K_1 is uniformized automatically, and with the help of ramification theory and quadratic transformation theory it is shown that because v_1 is uniformizable so also is v^*, an extension of v_1 to K^* that is also an extension of v. As a final step, the uniformizability of v is deduced from that of v^*. In the last chapter, varieties are defined as aggregates of local domains, the existence of finite resolving systems in the sense of Zariski is proved for two-dimensional function fields, and finally the existence of non-singular models for such fields is demonstrated. This chapter is rather short, and reference is made to Zariski's papers for many of the proofs.

H. T. Muhly (Iowa City, Iowa)

4159:

Samuel, Pierre. Multiplicités de certaines composantes singulières. Illinois J. Math. **3** (1959), 319–327.

Let V and V' be normal varieties and let f be a birational mapping of V' to V that is everywhere regular on V'. The paper is concerned with the problem of formulating a proper definition of $f^{-1}(D)$, where D is a divisor such that the support of D on V contains singular points of V that are fundamental elements of f^{-1}. Thus if W' is a subvariety of codimension one on V', then for each divisor D on V it is required to define the coefficient $\overline{w}(D)$ of W' in $f^{-1}(D)$. If w is the valuation of the function field K of V' that is defined by W', if $\mathfrak{o}$ is the local ring of $W = f(W')$, and if $v_\mathfrak{p}$ is the valuation of K associated with the minimal prime ideal $\mathfrak{p}$ of $\mathfrak{o}$, this problem gives rise to the following question of existence. Does there exist a function, $\mathfrak{p} \to \overline{w}(\mathfrak{p})$, on the set I of prime ideals of rank one in $\mathfrak{o}$ to the non-negative reals, of such a nature that $w(x) = \sum_\mathfrak{p} \overline{w}(\mathfrak{p}) v_\mathfrak{p}(x)$ for all non-zero $x \in K$? This question is answered in the affirmative for any noetherian local domain $\mathfrak{o}$ that is integrally closed in its quotient field K and for any discrete rank one valuation w of K with a valuation ring that dominates $\mathfrak{o}$. If $w'(\mathfrak{p}) = \inf \{w(x)/v_\mathfrak{p}(x);\ x \in \mathfrak{p},\ x \neq 0\}$, and if $\overline{w}(\mathfrak{p})$ is any solution of the existence problem, then $\overline{w}(\mathfrak{p}) \leqq w'(\mathfrak{p})$. Moreover, if w' is a solution to the problem it is the only solution, and if the problem has a unique solution it must be w'. A minimal prime ideal $\mathfrak{p}$ of $\mathfrak{o}$ is called almost principal relative to w if for each $\varepsilon > 0$ there is an element $x \in \mathfrak{p}$ $(x \neq 0)$ such that

$$\sum_{\mathfrak{q} \neq \mathfrak{p}} w'(\mathfrak{q}) v_\mathfrak{q}(x) \leqq \varepsilon v_\mathfrak{p}(x),$$

and o is said to be almost Gaussian (relative to w) in case each minimal prime ideal of o is almost principal. It is shown that a necessary and sufficient condition that the above-mentioned existence problem have a unique solution is that o be almost Gaussian relative to w. Finally it is shown that if o is the local ring associated with the vertex of the cone that projects a non-singular plane curve C, then o is almost Gaussian relative to any discrete rank one valuation w that dominates o. On the other hand, the local ring associated with the vertex of a three-dimensional cone with a quadric surface as base provides an example in which there is more than one solution to the existence problem. *H. T. Muhly* (Iowa City, Iowa)

4160:

Nishi, Mico. Some results on abelian varieties. Nat. Sci. Rep. Ochanomizu Univ. **9** (1958), 1–12.

Let A^d be an abelian variety, and $\sim$ denote linear equivalence of divisors. A divisor X on A is called non-degenerate if the subgroup of all $t \in A$ such that $X_t \sim X$ is finite. If X is also positive, then we have $l(X) = X^{(d)}/d! = (-1)^{d-1}\chi_A(X)$ ($X^{(d)}$ d-fold intersection number of X with itself, $\chi_A(X)$ virtual arithmetic genus of X), and $l(aX) = a^d l(X)$ for all integers $a \geqq 0$. Imbedding A in projective space, the Hilbert characteristic function of A is (order A)$m^d/d!$ and the arithmetic genus of A is 0. For any divisor X on A, we have $\chi_A(-X) = (-1)^d\chi_A(X)$, $\chi_A(X) = (-1)^{d-1}X^{(d)}/d!$ and $\chi_A(mX) = m^d\chi_A(X)$. A divisor X is degenerate if and only if $\chi_A(X) = 0$. There is a discussion of degenerate divisors and a direct proof of the fact that $G(A)/G_n(A)$ is a free abelian group of finite rank. *P. Samuel* (Urbana, Ill.)

4161:

★Cartier, Pierre. Dualité des variétés abéliennes. Séminaire Bourbaki; 10e année: 1957/1958. Textes des conférences; Exposés 152 à 168; 2e éd. corrigée, Exposé 164, 13 pp. Secrétariat mathématique, Paris, 1958. 189 pp. (mimeographed)

Apart from some expository material, this contains a criterium for a divisor class C (on a complete non-singular variety X) to contain a divisor, rational over a field k, when it is known that C contains a divisor, rational over some field k' between k and $k^{1/p}$ (p = characteristic). This is applied first to the proof of the following theorem: if X, defined over an algebraically closed field k, is complete and non-singular, there is a canonical homomorphism, injective in general and bijective if X is an abelian variety, of the Lie algebra of the Picard variety of X into $H^1(X, \mathcal{O}_k)$, where $\mathcal{O}_k$ is the sheaf of local rings on X. As a consequence, the author sketches briefly a proof for the following facts: if A and B are abelian varieties, and α is an isogeny of A onto B, the transpose of α has the same degree as α; in particular, every abelian variety is reflexive (i.e. canonically isomorphic to the dual of its dual). Proofs for the same results have been published in the meanwhile by M. Nishi, Mem. Coll. Sci. Univ. Kyoto Ser. A. Math. **32** (1959), 333–350. Another proof, superseding the one sketched in the paper under review, has appeared in the author's paper [Ann. of Math. (2) **71** (1960), 315–351]. *A. Weil* (Princeton, N.J.)

4162:

★Tate, J. *WC*-groups over $\mathfrak{p}$-adic fields. Séminaire Bourbaki; 10e année: 1957/1958. Textes des conférences; Exposés 152 à 168; 2e éd. corrigée, Exposé 156, 13 pp. Secrétariat mathématique, Paris, 1958. 189 pp. (mimeographed)

Let k be a $\mathfrak{p}$-adic number field, K the algebraic closure of k, and G the Galois group of K/k. Let A be an abelian variety defined over k, and A_K the group of points of A rational over K. Let $WC(A/k)$ denote the one-dimensional cohomology group $H^1(G, A_K)$ constructed with cochains coming from finite extensions of k in K. The main result of the paper then says that the discrete abelian group $WC(A/k)$ is canonically dual to the compact abelian group B_k of the points rational over k on the Picard variety B of A. For the proof, a canonical pairing of $WC(A/k)$ and B_k into the Brauer group $\mathrm{Br}(k)$ is defined by geometric method, and then a pairing of the same groups into the reals mod. 1 by using local class field theory; the latter is then proved to be a dual pairing of discrete $WC(A/k)$ and compact B_k.

As an application, the following result is also stated: If Γ is an algebraic curve defined over, and with rational point in, k, and if L/k is a finite Galois extension with group S, then $H^1(S, C_L) = 0$, where C_L is the group of idèle classes of the field of rational functions on Γ defined over L. *K. Iwasawa* (Cambridge, Mass.)

LINEAR ALGEBRA

See also 4122, 4213, 4228.

4163:

Goldbaum, Ia. S. On a transformation of the secular equation. J. Appl. Math. Mech. **22** (1958), 750–753 (539–541 Prikl. Mat. Meh.).

A detailed but elementary proof is given of the well-known result that the roots of the determinantal equation

$$\begin{vmatrix} 1 & \lambda & \lambda^2 & \cdots & \lambda^n \\ x & Ax & A^2x & \cdots & A^nx \end{vmatrix} = 0,$$

where A is an $n \times n$ matrix, x a vector, are the latent roots of the matrix A. A few other relationships concerning this determinant are also obtained. This same equation is considered in a broader perspective in a recent paper by A. S. Householder and F. L. Bauer [Numer. Math. 1 (1959), 29–37; MR **20** #7387]. *B. A. Chartres* (Sydney)

4164:

Kasajima, Tomomi. A note on a theorem of Rutishauser. Comment. Math. Univ. St. Paul. **6** (1957), 89–91.

The author states and proves the following slightly more general form of the LR-algorithm of Rutishauser [C. R. Acad. Sci. Paris **240** (1955), 34–36; MR **16**, 785]. Let A be a real symmetric positive-semidefinite matrix. Let $A_0 = A$, $A_k = L_kL_k'$, $A_{k+1} = L_k'L_k$, where the prime denotes transposition ($k = 0, 1, 2, \cdots$). Then $\lim_k A_k$ exists and is a diagonal matrix whose diagonal elements are the eigenvalues of A in some order. The proof is short and straightforward. {Reviewer's note: Rutishauser

proved the above theorem in a paper apparently submitted in 1956 but not published until 1958 [Theorem 4 of Nat. Bur. Standards Appl. Math. Ser. no. 49 (1958), 47–81; MR **19**, 770].} *G. E. Forsythe* (Stanford, Calif.)

4165:

Fan, Ky; and Householder, A. S. A note concerning positive matrices and M-matrices. Monatsh. Math. **63** (1959), 265–270.

A real square matrix is called an M-matrix if (i) it is non-singular; (ii) all its off-diagonal elements are non-positive; (iii) all elements of its inverse are non-negative. Let A be a positive matrix and B an M-matrix, both of type $n \times n$, where $n \geq 2$. A classical theorem of Perron implies that there exists a unique number λ and a vector x (unique up to a positive scalar multiple) such that $\lambda > 0$, $x > 0$, $Ax = \lambda Bx$. For $1 \leq k \leq n$, denote by A_k the matrix obtained when all elements in the kth row and the kth column of A are replaced by zeros; and let B_k be defined similarly. Since every principal submatrix of B is an M-matrix, there exists a unique number λ_k and a vector $x^{(k)} = (x_1^{(k)}, \cdots, x_n^{(k)})$ (unique up to a positive scalar multiple) such that

$$\lambda_k > 0;\ x_i^{(k)} > 0\ (i \neq k);\ x_k^{(k)} = 0;\ A_k x^{(k)} = \lambda_k B_k x^{(k)}.$$

It is shown that (a) $\lambda > \lambda_k$ $(1 \leq k \leq n)$; (b) there exist n positive numbers c_k such that $x = c_1 x^{(1)} + \cdots + c_n x^{(n)}$; (c) for any fixed k, a positive number ρ satisfies the inequality $\lambda > \rho > \lambda_k$ if and only if $A - \rho B$ is non-singular and all elements in the kth row and kth column of $(A - \rho B)^{-1}$ are positive. Two different proofs of (b) are given; one algebraic, the other topological. Special cases of (a) and (c) were known previously; in particular, the case $B = I$ of (c) had been given by the authors in earlier papers [K. Fan, Monatsh. Math. **62** (1958), 219–237; MR **20** #2354; A. S. Householder, ibid. **62** (1958), 238–242; MR **20** #2355]. *L. Mirsky* (Sheffield)

4166:

Marcus, M.; and Khan, N. A. A note on the Hadamard product. Canad. Math. Bull. **2** (1959), 81–83.

Let $A = (a_{ij})$ and $B = (b_{ij})$ be two complex $n \times n$ matrices, and denote by $H = (h_{ij})$ the 'Hadamard product' of A and B, so that $h_{ij} = a_{ij} b_{ij}$. Further, let K be the Kronecker product of A and B. It is shown that H is a principal submatrix of K and from this it is deduced that, if A and B are non-negative (positive definite) hermitian, then so is H, and $\lambda_{s+n^2-n} \leq \mu_s \leq \lambda_s$ $(1 \leq s \leq n)$, where $\lambda_1 \geq \cdots \geq \lambda_{n^2}$ are the characteristic roots of K and $\mu_1 \geq \cdots \geq \mu_n$ those of H. As a further application of the method, an inequality relating to Hadamard products of real non-negative matrices is proved. *L. Mirsky* (Sheffield)

4167:

Marcus, Marvin; and Purves, Roger. Linear transformations on algebras of matrices: the invariance of the elementary symmetric functions. Canad. J. Math. **11** (1959), 383–396.

Let M_n be the set of all $n \times n$ matrices. For $A \in M_n$, denote by $E_r(A)$ the rth elementary symmetric function of the characteristic roots of A. The principal result established in the present paper is as follows. If $4 \leq r \leq n-1$, if T is a linear transformation of M_n into itself, and if $E_r(T(A)) = E_r(A)$ for all $A \in M_n$, then T is essentially a similarity transformation; more precisely, there exist matrices U, V and a number λ such that $UV = \lambda I$, $\lambda^r = 1$ and, for all $A \in M_n$, $T(A) = UAV$ or $T(A) = UA'V$. Further, this result is shown to be false for $r = 1, 2$, while its truth for $r = 3$ remains an open question. The case $r = n$ had been discussed earlier by M. Marcus and B. N. Moyls [Canad. J. Math. **11** (1959), 61–66; MR **20** #6432]. *L. Mirsky* (Sheffield)

4168:

Bergmann, Artur. Ein Axiomensystem für Determinanten. Arch. Math. **10** (1959), 243–256.

The author gives a new intrinsic characterization of determinants. Let V be a vector space of dimension n over a commutative integral domain R with unit element. A norm function on $\mathrm{Hom}_R(V, V)$ is a non-zero homomorphism N of the multiplicative semigroup of $\mathrm{Hom}_R(V, V)$ into R. Theorem: There exists a norm function N on $\mathrm{Hom}_R(V, V)$ satisfying the properties

(I) $$N(xy) = N(x) \cdot N(y);$$

(II) $$n![x_1, \cdots, x_{i-1}, \xi_i x_i + \eta_i y_i, x_{i+1}, \cdots, x_n] = \xi_i n![x_1, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n] + \eta_i n![x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n],$$

where $i = 1, 2, \cdots, n$; $x_j, y_j \in \mathrm{Hom}_R(V, V)$; $\xi_i, \eta_i \in R$ and $n > p$ if R has characteristic $p > 0$; (III) $N(mE) = m^n \cdot 1$, where E is the unit element of $\mathrm{Hom}_R(V, V)$, m is a natural number, and 1 is the unit element of R. The function N is determined uniquely by these properties, and $N(x) = \det(x)$. The bracket symbol in (II) is defined recursively by $[x] = N(x)$ and, for $k \geq 1$,

$$(k+1)![x_1, \cdots, x_{k-1}, x_k, x_{k+1}] = k![x_1, \cdots, x_{k-1}, x_k + x_{k+1}] - k![x_1, \cdots, x_{k-1}, x_k] - k![x_1, \cdots, x_{k-1}, x_{k+1}].$$

The proof is extremely involved. As a definition of determinants it is unlikely to supersede the method of Bourbaki. However this, presumably, was not the intention. The main interest of this new characterization lies in its connection with the author's earlier work on norm functions [Math. Nachr. **15** (1956), 55–76; **19** (1958), 237–254; MR **18**, 109; **21** #3465] and the applications to algebras occurring in the theory of complex multiplication. The author also gives a similar intrinsic definition of the natural trace function for $\mathrm{Hom}_R(V, V)$ in the case where R is a commutative field.

W. E. Jenner (Lewisburg, Pa.)

ASSOCIATIVE RINGS AND ALGEBRAS

See also 4168, 4176.

4169:

Smiley, M. F. Remarks on the commutativity of rings. Proc. Amer. Math. Soc. **10** (1959), 466–470.

Soit R un anneau. Herstein a établit que si pour tous $x, y \in R$, $c = [x, y] = xy - yx$ satisfait à la relation $C^n = C$

pour un $n=n(c)\geq 2$, alors R est commutatif [Canad. J. Math. **9** (1957), 583–586; MR **19**, 1035]. L'auteur améliore ce résultat en utilisant la notion de degré k des commutateurs, posant $L_0(R)=R$, $L_{k+1}(R)=\{[x, y] \text{ avec } x\in R \text{ et } y\in L_k(R)\}$, k entier positif ou nul. Il établit que si pour tout entier $k\geq 1$ on a $x^n=x$ dès que $x\in L_k(R)$ $(n\geq 2)$ alors $L_k(R)=0$. Il s'ensuit que R est commutatif dès que l'on est dans l'un des cas suivants: (a) on a $k=1$ (cas traité par Herstein); (b) R ne possède aucun idéal nilpotent autre que zéro. La démonstration est complète et se ramène au cas des anneaux de division et des anneaux premiers au moyen du lemme suivant: si R satisfait à la condition (b), $a\in R$, $k\geq 1$ et $[a, t]=0$ pour tout $t\in L_k(R)$ entraîne que a est dans le centre de l'anneau R. On établit aussi à l'aide du lemme précédent que pour tout anneau R satisfaisant à (b) et possédant un anti-endomorphisme $a\to a'$ tel que $[a, a']=0$ pour tout $a\in R$, chaque élément $x\in R$ satisfait à une équation $x^2=cx+d$ avec $c, d\in Z$, Z désignant le centre de R.

J. Guérindon (Rennes)

4170:

Henriksen, Melvin. The $a^{n(a)}=a$ theorem for semirings. Math. Japon. **5** (1958/59), 21–24.

We shall say that a semiring R satisfies the condition (A) if to every $a\in R$ there is an integer $n(a)$ such that $a^{n(a)}=a$. A well-known result of Jacobson says that a ring satisfying condition (A) is necessarily commutative. This result is generalized as follows: Let R be a semiring with more than one element containing a zero element and an identity element e, and satisfying the condition (A). Suppose that e is the only non-zero idempotent $\in R$ and $a+e=e+a$. Then R is either a (commutative) field or a two-element lattice.

Several instructive examples are given. They serve to illustrate the difficulties encountered in attempting to generalize Jacobson's theorem to a wider class of semirings than those introduced by the author.

Št. Schwarz (Bratislava)

4171:

Lesieur, Léonce; et Croisot, Robert. Structure des anneaux premiers noethériens à gauche. C. R. Acad. Sci. Paris **248** (1959), 2545–2547.

Let A be a prime ring in which the ACC for left ideals is satisfied. A left ideal X of A is closed if for all b not in X there is an x in the left ideal $(b|$ generated by b such that $x\neq 0$ and $X\cap(x|=0$. The following results are announced. (1) The closed left ideals of A form a complemented modular lattice of finite length. (2) The closed left ideals coincide with the complement left ideals introduced by A. W. Goldie [Proc. Nat. Acad. Sci. U.S.A. **44** (1958), 584–586; MR **20** #3182]. (3) The maximal closed left ideals (distinct from A) coincide with the $\cap$-irreducible components of 0. (4) A necessary and sufficient condition that A be an integral domain is that 0 be $\cap$-irreducible as a left ideal. (5) If D' is the set of elements of A that are not right zero divisors, then A has a left quotient ring $Q(A)$ consisting of elements of the form $b^{-1}a$ where $a\in A$, $b\in D'$. The quotient ring $Q(A)$ is isomorphic to the ring of all square matrices of order n over a skewfield K, where n is the length of the lattice of closed left ideals. Moreover, Q is the injective envelope of A.

D. C. Murdoch (Vancouver, B.C.)

4172:

Tominaga, Hisao. A note on Galois theory of primary rings. Math. J. Okayama Univ. **8** (1958), 117–123.

In this article Nakayama's Galois theory for primary rings [Amer. J. Math. **77** (1955), 1–16; MR **16**, 560] and a theorem of F. Kasch are applied to establish a normal basis for certain (too complicated to describe here) Galois primary ring extensions. Almost all of the results are extensions to primary rings of theorems on the normal basis of simple rings (and the structure of certain group rings over simple rings) of the reviewer [Proc. Amer. Math. Soc. **9** (1958), 222–229; MR **20** #2356; cf. also Onodera and Tominaga, Math. J. Okayama Univ. **7** (1957), 77–81; MR **20** #2357]. *C. C. Faith* (Heidelberg)

4173:

Nagahara, Takashi. On generating elements of Galois extensions of division rings. IV. Math. J. Okayama Univ. **8** (1958), 181–188.

Let K be a division ring, and L a division subring; $V_K(S)$ is the centralizer in K of any (nonempty) subset S of K. A subring D is called simple over L if D is the division subring generated by L and a single element a in K; K/L is locally simple (l.s.) if every intermediate division ring D having finite dimension over L is simple over L. In previous papers [Nagahara, same J. **6** (1957), 181–190; MR **19**, 382; Moriya and Nagahara, ibid. **7** (1957), 89–94; MR **21** #1982; Nagahara, ibid., 173–178; MR **21** #1983], for the case K/L is finite and Galois, the author obtained conditions (e.g., $V_K(L)=V$ is commutative) for which K/L is l.s. In the present article these results are sharpened, and extended to the case K/L is locally finite-dimensional and Galois. Then, for example, K/L is l.s. if L has infinite dimension over $Z=L\cap V$, or, more generally, if $[L:Z]\geq n_0=\max\{n(W/Z)\}$, where $n(W/Z)$ is the minimal number of generators of W/Z, and W ranges over the intermediate (division) rings of V/Z having finite dimension. Here $n_0=\infty$ if no max. exists. If $n_0<\infty$, and if $n(D/L)$ is defined for the extension D/L similarly as $n(W/Z)$, then $n(D/L)\leq n_0$ when $[D:L]<\infty$, a result which shows that K/L is also l.s. if $n_0=1$, that is, if V/Z is l.s. This latter condition holds, for example, when V is commutative, since, under the assumptions, V is then algebraic and separable over Z. Other similar results are obtained. *C. C. Faith* (Heidelberg)

4174:

Cohn, P. M. On a class of simple rings. Mathematika **5** (1958), 103–117.

The central problem attacked is the solvability of the equation $a\xi-\xi b=c$ in an associative algebra R. The main results are these: Suppose (i) the equation solvable for each a, b, c, with neither a nor b zero; then R is simple, has no zero divisors and no identity. Conversely every algebra without zero divisors or an identity is embeddable in a ring satisfying (i). Variations on the equation condition are discussed and the corresponding classes of algebras related. The most difficult point in the investigation is the imbedding theorem. It is established by adjoining an identity, constructing the non-commutative transcendental extension by an element x, and then factoring by the ideal generated by $ax-xb-c$ for some choice of a, b, c in R. This provides the basis for an

inductive process by which solutions for all the equations are included in a single extension. Repetition leads to an ascending sequence of algebras whose union has the required properties. *W. G. Lister* (Oyster Bay, N.Y.)

4175:

Green, H. F. Isomorphic rings of infinite matrices. Quart. J. Math. Oxford Ser. (2) **10** (1959), 38–42.

The background of this paper, together with definitions and notations employed, is to be found in the reviewer's *Infinite matrices and sequence spaces* [Macmillan, London, 1950; MR **12**, 694] and *Linear operators* [Macmillan, London, 1953; MR **15**, 719], referred to below as "I.M." and "L.O." respectively. Throughout α, β denote sequence spaces, ϕ is the space of all finite sequences (I.M., 273), and α^* is the dual space of α (I.M., 275). The following results are proved.

Theorem 1: If α and β contain ϕ and are homoeomorphic (L.O., 291), there exists a matrix A such that $\beta = A\alpha$, and A has a unique two-sided reciprocal A^{-1}, which maps β on α. Theorem 2: If (i) $\alpha \supseteq \phi$, $\beta \supseteq \phi$, (ii) $\phi \subseteq \alpha' \subseteq \alpha^*$, $\phi \subseteq \beta' \subseteq \beta^*$, (iii) g is an isomorphism over the centre (L.O., 321, 322) of the ring $\Sigma_{\alpha'}(\alpha)$ on the ring $\Sigma_{\beta'}(\beta)$ (L.O., 298, 307), then there exists a matrix A with a two-sided reciprocal A^{-1} such that (i) $A\alpha = \beta$, $A'\beta' = \alpha'$, A' being the transpose of A; (ii) $A^{-1}\beta = \alpha$, $(A^{-1})'\alpha' = \beta'$; (iii) $g(X) = AXA^{-1}$ for all $X \in \Sigma_{\alpha'}(\alpha)$. Theorem 3: If (i) $\alpha \supseteq \phi$, $\beta \supseteq \phi$, (ii) $\Sigma(\alpha)$ and $\Sigma(\beta)$ are rings which are isomorphic over the centre, then α and β are homoeomorphic, and so are α^* and β^* [cf. L.O., 323]. Theorem 4: If α and β contain ϕ and are normal (I.M., 278), the rings $\Sigma(\alpha)$ and $\Sigma(\beta)$ are isomorphic over the centre if and only if α and β are homoeomorphic. Theorem 5: If (i) α and β contain ϕ, (ii) $\Sigma(\alpha)$ and $\Sigma(\beta)$ are rings which are isomorphic over the centre, then the rings $\Sigma(\alpha^{**})$ and $\Sigma(\beta^{**})$ are isomorphic over the centre.

The essential step in the proof of these results follows as a particular case of a theorem of N. Jacobson [*Lectures in abstract algebra*, vol. II, Van Nostrand, Toronto-New York-London, 1953; MR **14**, 837; p. 266].

R. G. Cooke (London)

HOMOLOGICAL ALGEBRA

See also 4190, 4424.

4176:

Rosenberg, Alex; and Zelinsky, Daniel. Finiteness of the injective hull. Math. Z. **70** (1958/59), 372–380.

Let R be a ring with unit element and N its radical. We shall say that a left R-module is of type F if it is embeddable in an injective left R-module of finite (composition) length. Suppose first that R satisfies the left minimum condition, and let A be a simple left R-module. If we denote by V_i, $i = 0, 1, 2, \cdots$, the homogeneous component of the left module N^i/N^{i+1} corresponding to A, then V_i is a two-sided R-module and so the ring R_i of mappings induced by the right operation of R on V_i is a semi-simple subring of the endomorphism ring E_i of the left module V_i, which is a simple ring. The fundamental result is that A is of type F if and only if E_i is finitely generated as a left R_i-module for $i = 0, 1, 2, \cdots$. An example is given for a ring with left minimum condition but not with right minimum condition and admitting no non-zero injective left-module. As for the question of existence of a ring satisfying both left and right minimum conditions but having a simple left-module not of type F, the authors show that it is equivalent to the well-known question of existence of a simple ring E and a simple subring T such that E is finitely generated as a right T-module but not as a left T-module. We now drop the minimum conditions. Then if every simple left R-module is of type F the radical is a nilideal and $\bigcap_{i=0}^{\infty} N^i = 0$, and if moreover R has only a finite number of simple left-modules, up to isomorphism, N must be nilpotent. The authors finally assume that R is commutative to prove for example that a simple R-module R/M with a maximal ideal M is of type F if and only if the corresponding ring of quotients R_M satisfies the minimum condition. As an application, a second proof is given to a theorem of Kaplansky: a commutative ring R is a regular ring if and only if every simple R-module is injective. *G. Azumaya* (Sapporo)

GROUPS AND GENERALIZATIONS

See also 4356, 4385, 4412.

4177:

Marchionna Tibiletti, Cesarina. Una scomposizione del prodotto sghembo di Rédei. Univ. e Politec. Torino. Rend. Sem. Mat. **17** (1957/58), 209–221.

The author continues and, in a certain sense, concludes the study of the skew product $G \circ \Gamma$ of two groups G and Γ that was initiated by Rédei [J. Reine Angew. Math. **188** (1950), 201–227; MR **14**, 13] and carried further by Kochendörffer [ibid. **192** (1953), 96–101; MR **15**, 852] and Rühs [ibid. **198** (1957), 81–86; MR **20** #900]. These authors have proved that all k-fold degenerate ($k = 4, 3, 2, 1$) skew products can be obtained by forming Schreier extensions of given groups or products of permutable subgroups intersecting trivially (Zappa-Szép extensions). The present author proves that also in the most general, the non-degenerate, case the structure of a skew product $G \circ \Gamma$ is as follows: G has a certain subgroup G_1 and Γ a subgroup Γ_1. One begins by forming a skew product $G \circ \Gamma_1$ which turns out to be a Schreier extension of Γ_1 by G and similarly $G_1 \circ \Gamma$, a Schreier extension of G_1 by Γ. These two groups are permutable and their intersection is the direct product of G_1 and Γ_1. The group $G \circ \Gamma$ is their product. To sum up: all possible skew products of two given groups G and Γ can be obtained from the original data by applying once or several times the operations of forming a Schreier extension of given groups, in particular the direct product, or of forming a group that is the product of two given permutable subgroups whose intersection is trivial or a known subgroup. *K. A. Hirsch* (London)

4178:

Moran, S. Associative operations on groups. III. Proc. London Math. Soc. (3) **9** (1959), 287–317.

The author continues his study of associative operations on groups [parts I and II: same Proc. (3) **6** (1956), 581–

596; 8 (1958), 548–568; MR 20 #3908, #7054]. Two new types of regular multiplications [see Golovin, Mat. Sb. (N.S.) 27 (69) (1950), 427–454; MR 12, 672] are defined and investigated for associativity; the same is done for the corresponding reduced multiplications [see part II]. Some theorems on subgroups of a free product are proved as part of the investigation of associativity.

$N(X)$ is called a normal subgroup function if it associates with every group G a normal subgroup $N(G)$ of G. Let F, G_α, $[G_\alpha]^F$ be defined as in the review of part I. Let $C_N[G_\alpha]^F = N(F) \cap [G_\alpha]^F$. Then the N-product of the G_α is defined to be $F/C_N[G_\alpha]^F$ and the reduced N-product is defined to be $F/N(F)$. Let $N^*(F) = \prod_\alpha N(G_\alpha)^F$, where $N(G_\alpha)^F$ is the normal subgroup in F generated by $N(G_\alpha)$. The N^* product of the G_α is defined to be $F/N^*(F) \cap [G_\alpha]^F$ and the reduced N^* product is defined to be $F/N^*(F)$. Sufficient conditions are given for the associativity of each of the four types of products. These criteria are applied to a large number of particular examples. Examples where associativity breaks down are given.

Products obtained by taking unions and intersections of normal subgroup functions are investigated. The intersection of verbal subgroups need not be a verbal subgroup [Neumann, Math. Ann. 114 (1937), 506–552] but the corresponding regular and reduced multiplications are associative.

A large number of cases in which the normal subgroup function of a proper free product (not the infinite dihedral group) is trivial are given. A proper free product (not the infinite dihedral group) has no non-trivial abelian normal subgroups. *R. R. Struik* (Vancouver, B.C.)

4179:

Baumslag, Gilbert. Wreath products and p-groups. Proc. Cambridge Philos. Soc. 55 (1959), 224–231.

The author uses wreath products (defined below) to prove several embedding theorems, and as a side result constructs (in a simple manner) some p-groups without center.

Let A, B be groups. Let A_b be an isomorphic copy of A indexed by $b \in B$. Let K be the direct product of A_b, $b \in B$. Then W, the wreath product of A by B, is $\mathrm{gp}(K, B; c^{-1}a_b c = a_{bc},\ a \in A,\ b, c \in B)$, where $\mathrm{gp}(X; R)$ denotes the group generated by the set X with defining relations R.

The author uses wreath products to prove that every group can be embedded in a divisible group, every p-group can be embedded in a divisible p-group, and every periodic group can be embedded in a divisible periodic group. A nilpotent group cannot always be embedded in a divisible nilpotent group [See S. N. Černikov, Mat. Sb. (N.S.) 18 (60) (1946), 397–422; MR 8, 311]. The author uses an unpublished example of B. H. Neumann to show that an R-group cannot always be embedded in a divisible R-group. (An R-group is a group in which extraction of roots, whenever possible, is unique.) B. H. Neumann [J. London Math. Soc. 18 (1943), 4–11; MR 5, 58] originally proved that every group can be embedded in a divisible group, but his method (free products with amalgamations) cannot be used to prove the other embedding theorems of the author.

In proving these theorems, the author states that Schmidt's example [Mat. Sb. (N.S.) 8 (50) (1940), 363–375; MR 2, 214] of a p-group without center is the wreath product of a cyclic group of order p by a $Z(p^\infty)$. The author also proves: W is nilpotent if and only if A and B are nilpotent p-groups (for the same prime p) with A of finite exponent and B finite. If G/A is a finite p-group and A is a nilpotent p-group of finite exponent, then G is nilpotent. *R. R. Struik* (Vancouver, B.C.)

4180:

Nash-Williams, C. St J. A. Abelian groups, graphs and generalized knights. Proc. Cambridge Philos. Soc. 55 (1959), 232–238.

Let ν be a cardinal number. A "chessboard" is the set of all such lattice points of ν-dimensional space, which have only finitely many non-zero coordinates. A piece in this chessboard is defined to be a generalized knight if every vector obtainable from a move of the piece by permuting its components and changing the signs of a subset of them is also a permitted move. As a generalization of a result of Kürschák [Math. Phys. Lapok 33 (1926), 117–119; Acta Sci. Math. Szeged 4 (1928–29), 12–13] and Vázsonyi [Acta Sci. Math. Szeged 9 (1938/40), 163–173; MR 1, 105] the main result of the paper gives a necessary and sufficient condition that, in the case $\nu \leq \aleph_0$, a generalized knight should trace out the whole chessboard in a one-ended and in an endless infinite sequence of moves respectively (visiting each position exactly once). The proofs proceed by group-theoretical and graph-theoretical methods. The following theorem is interesting also in itself: if g is a set of generators of an enumerably infinite abelian group A, then the elements of A can be arranged in both a one-ended and an endless infinite sequence in which successive terms differ by $\pm$ an element of g, except that the one-ended arrangement is impossible if g is finite and the rank of A is 1. *A. Kertész* (Debrecen)

4181:

Walker, Elbert A. Subdirect sums and infinite Abelian groups. Pacific J. Math. 9 (1959), 287–291.

Let G be a torsion-free abelian group of infinite rank. Then G is isomorphic to a subgroup G' of a discrete direct sum of copies H_α of the rational numbers such that G' intersects each H_α non-trivially and the projection of G' into H_α is onto H_α for each α. *L. Fuchs* (Budapest)

4182:

Borevič, Z. I. On the proof of the principal ideal theorem. Vestnik Leningrad. Univ. 12 (1957), no. 13, 5–8. (Russian. English summary)

The item MR 21 #68 was printed in the wrong section; it concerns the group-theoretic theorem, used in algebraic number theory, that if H is a subgroup of finite index in a group G, then the "transfer" homomorphism $V: G \to H/H'$ is identically 0 when G is finitely generated and $H \subset G'$. Here $'$ denotes commutator subgroup, and V is given by $V(x) = (\prod_\rho \rho x \rho x^{-1})H'$, where ρ runs over a set of representatives of right cosets of H.

4183:

Hall, P. Some sufficient conditions for a group to be nilpotent. Illinois J. Math. 2 (1958), 787–801.

Let $G = G_0 > G_1 > \cdots > G_m = 1$ be a chain of subgroups

of a group G. Collect all the automorphisms α that leave every coset of G_i in G_{i-1} fixed ($i=1, 2, \cdots, m$). They form a group A, the "stability" group of the given chain. Kaloujnine [*Bericht über die Mathematiker-Tagung in Berlin, 1953*, pp. 164–172, Deutscher Verlag der Wiss., Berlin, 1953; MR **17**, 456] has proved that A is always soluble and, if the G_i are assumed to be normal in G, even nilpotent of a class not exceeding $m-1$. This bound on the class of A is actually assumed, for example when G is an elementary abelian group of order p^m: A is then a Sylow p-subgroup of the automorphism group of G. It is one of the main results of the present paper that even in the general case A is nilpotent, of a class not exceeding $\frac{1}{2}m(m-1)$. This result appears in an equivalent form which makes use of the bracketless notation for commutator subgroups: Let $H, K, L, \cdots$ be subgroups of a group G and set $\gamma HK=[H, K]$, $\gamma^2HKL=[[H, K], L]$, $\gamma^nHK^n=[\cdots[[H, K], K], \cdots, K]$ (n K's), etc. The author's theorem is then: If H and K are subgroups of a group and $\gamma^mHK^m=1$, then $\gamma^{n+1}K^{n+1}H=1$, where $n=\frac{1}{2}m(m-1)$. The above main theorem now arises by taking for H our group G, in its regular representation, for K our A and for the containing group the holomorph of G. Whether the least upper bound $c(m)$ of the class of A is, in fact, $\frac{1}{2}m(m-1)$, remains an open question. But for $m=3$ the author constructs an example (of surprising complexity) to show that $c(m)=3$, so that there is a genuine difference between the cases of normal G_i and arbitrary G_i.

Now let A be any group of automorphisms of G and suppose that $G>[G, A]=\gamma GA>\gamma^2GA^2>\cdots>\gamma^mGA^m=1$. Kaloujnine has shown [loc. cit.] that if all γ^iGA^i are normal in G, then γGA is nilpotent. In the present paper the author proves that in the general case γGA is at least locally nilpotent; but he gives examples to show that the group γGA need not be nilpotent and that, even for finite G, there is no bound on the class of γGA in terms of m, except that for $m=2$ the group is always abelian. Again, the theorem is established in a somewhat wider setting: Let H and K be subgroups of a group G and suppose that $\gamma^mHK^m=1$. Let $H_1=\gamma HK$ and let $\bar{H}=HH_1$ and $K_1=KH_1$ be the normal closures of H and K, respectively, in the group $\{H, K\}$. Let C be the centralizer of $\bar{H}$ in K_1. Then the groups K_1/C and H_1 are locally nilpotent.

Finally the author proves that if M is a nilpotent group and M' its derived group, and if G is an extension of M' by a nilpotent group, then G is itself nilpotent. More explicitly: if M is of class c and G/M' of class d, then G is of a class not exceeding $d\binom{c+1}{2}-\binom{c}{2}$.

K. A. Hirsch (London)

4184:

Inagaki, Nobuo. On a group whose commutator subgroup is nilpotent. Sci. Rep. Tokyo Kyoiku Daigaku Sect. A **6** (1958), 143–146.

Let G be a finite group with the property of the title. The author obtains an upper bound for the rank of G (largest exponent to which any prime occurs in the orders of the principal factors) under certain conditions on the Sylow subgroups of G and the minimal normal subgroups of G/Φ (Φ the Frattini subgroup). In particular, if the minimal normal subgroups of G/Φ are all of prime order, then rank $(G)=1$, G is supersolvable.

P. J. Higgins (London)

4185:

Curzio, Mario. Sugli N-gruppi risolubili. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. **25** (1958), 447–452.

An N-group is defined to be a finite group in which every normal subgroup is the only one of its order. The author proves (Theorem I) that the finite soluble group G is an N-group if, and only if, it satisfies the following three conditions: (1) Its Sylow subgroups are cyclic or elementary abelian or quaternion; (2) if G has a quaternion subgroup, it has no normal subgroups of order $4m$, where m is odd; (3) the order of every elementary abelian Sylow subgroup appears as the order of a principal factor of G. Next (Theorem II), the finite group G is a supersoluble N-group if, and only if, all its Sylow subgroups are cyclic. This fact uses and complements a theorem of K. Honda [Comment. Math. Univ. St Paul. **1** (1952), 5–39; MR **14**, 1059]. Theorem III: A finite soluble group G is an N-group if it is the product of two N-groups of coprime orders. Finally some simple consequences of these theorems are derived, of which the following is typical: The Frattini subgroup of the soluble N-group G is trivial if, and only if, every minimal normal subgroup of G is a Sylow subgroup.

B. H. Neumann (Manchester)

4186:

Baer, Reinhold. Verstreute Untergruppen endlicher Gruppen. Arch. Math. **9** (1958), 7–17.

A Hall subgroup H of a finite group G is one whose index and order are relatively prime. Let U be a subgroup of G whose order $o(U)$ divides $o(H)$. Wielandt has shown that if H is nilpotent, then U is conjugate in G to a subgroup of H. P. Hall has shown that the nilpotency of H cannot be replaced by supersolvability. The author shows that if H is a Hall σ-group of a σ-regular group G, then Wielandt's conclusion follows. He defines a partial ordering σ in a set of primes σ, and defines a segment s of σ to be a subset that includes with each prime q all "preceding" primes p such that $p\sigma q$. He calls a group G s-closed if its s-elements form a characteristic subgroup; σ-dispersed if it is s-closed for each segment s of σ; and a group of characteristic σ (or σ-group) if σ is (or includes) the set of prime divisors of $o(G)$. Then G is called σ-regular if (a) G possesses a σ-dispersed Hall σ-subgroup of characteristic σ; (b) each subgroup U contains a σ-subgroup V whose index is not divisible by any prime of σ; and (c) the condition $p\sigma q$ implies that no element of G induces an automorphism of order p in a q-subgroup of G. Other theorems involving σ-dispersion precede this main theorem. For example, a σ-group G is σ-dispersed if and only if $q\sigma p$ implies that some element of G induces an automorphism of order $p\neq q$ in a q-subgroup of G.

J. S. Frame (East Lansing, Mich.)

4187:

Baer, Reinhold. Supersoluble immersion. Canad. J. Math. **11** (1959), 353–369.

The normal subgroup K of the finite group G is said to be supersolubly immersed if the chief-factors of G below K are cyclic. It is proved that K is supersolubly immersed in G if $K/\phi K$ is supersolubly immersed in $G/\phi K$. Also, call the finite group G strictly p-closed if it has a normal p-subgroup N such that G/N is abelian and of exponent $p-1$. Then the normal subgroup K of the finite group G is

supersolubly immersed in G if and only if, for all primes p, and for all p-subgroups U of K, NU/CU is strictly p-closed, where NU and CU are the normaliser and centraliser of U in G. *Graham Higman* (Oxford)

4188:

Tarţia, Cornel. Sur les polynômes abstraits définis par groupes cycliques d'ordre fini et avec les valeurs en groupes cycliques d'ordre fini. Acad. R. P. Romîne. Fil. Cluj. Stud. Cerc. Mat. **8** (1957), 181–187. (Romanian. Russian and French summaries)

"Cette note présente les premiers résultats sur l'existence et le nombre des polynômes abstraits définis par groupes cycliques d'ordre fini et avec les valeurs en groupes cycliques d'ordre fini. On considère seulement le cas où l'un des groupes G ou G' est du deuxième ordre."

Résumé de l'auteur

4189:

Klingen, Helmut. Bemerkung über Kongruenzuntergruppen der Modulgruppe n-ten Grades. Arch. Math. **10** (1959), 113–122.

Denote by Γ the modular group of matrices $\begin{pmatrix} ab \\ cd \end{pmatrix}$ with rational integral entries and $ad-bc=1$. If ρ is an arbitrary integer, then $\Gamma_0(\rho)$, $\Gamma^0(\rho)$, $\Gamma_0{}^0(\rho)$ and $\Gamma(\rho)$ stand for the subgroups of Γ whose elements satisfy (mod ρ) the conditions $c\equiv 0$; $b\equiv 0$; $c\equiv b\equiv 0$; and $b\equiv c\equiv 0$, $a\equiv d\equiv 1$, respectively. The algebraic structure of Γ has been determined by F. Klein [*Gesammelte mathematische Abhandlungen*, v. 3, Springer, Berlin, 1923]. For ρ a rational prime, the corresponding problem for $\Gamma_0(\rho)$ (hence, also for $\Gamma^0(\rho)$, its conjugate) has been solved by H. Rademacher [Abh. Math. Sem. Univ. Hamburg **7** (1929), 134–148]; for $\Gamma(\rho)$ by H. Frasch [Math. Ann. **108** (1933), 229–252] and for $\Gamma_0{}^0(\rho)$ by the reviewer [Amer. J. Math. **72** (1950), 809–834; MR **12**, 591]. The situation can be generalized as follows. Let the field K be a finite algebraic extension over the rationals, denote by S_n the symplectic group of square matrices of order $2n$ and let ρ be an integral ideal in K. Then $\mathfrak{M}_n$, the modular group of degree n in K, consists of the matrices $M\in S_n$, $M=\begin{pmatrix} AB \\ CD \end{pmatrix}$, where A, B, C, D are square matrices of order n with elements in K. If E stands for the unit matrix, define $\mathfrak{M}_n(\rho)$, the principal congruence subgroup mod ρ, by the condition $M\equiv E$ (mod ρ), and, similarly, the congruence subgroup $\mathfrak{G}_n(\rho)$ by $C\equiv 0$ (mod ρ). These congruence subgroups have been considered by M. Koecher [Math. Z. **59** (1954), 399–416; MR **15**, 603], and the present author has determined a system of generators for $\mathfrak{M}_n$ [Nachr. Akad. Wiss. Göttingen, Math.-Phys. Kl. IIa **1956**, 173–185; MR **18**, 191]. In the present paper he solves the corresponding problem for $\mathfrak{M}_n(\rho)$ and $\mathfrak{G}_n(\rho)$; in addition, he also determines the indices of these subgroups in $\mathfrak{M}_n$ and obtains a necessary and sufficient condition for a $2n$ by m ($m\leq n$) matrix to permit its completion to an element of either $\mathfrak{M}_n(\rho)$ or $\mathfrak{G}_n(\rho)$. The method used is mainly induction on n, starting from the (classical) case $n=1$. The (finite) sets of generators of $\mathfrak{M}_n(\rho)$ and $\mathfrak{G}_n(\rho)$ obtained are too complicated for quotation here; the index $[\mathfrak{M}_n:\mathfrak{M}_n(\rho)]=\mathfrak{N}(\rho)^{n(2n+1)}\prod_{\pi\mid\rho}\prod_{\nu=1}^{n}(1-1/\mathfrak{N}(\pi^{2\nu}))$ ($\mathfrak{N}(\rho)$ = norm of ρ, π = prime ideal) and a similar formula holds for $[\mathfrak{M}_n:\mathfrak{G}_n(\rho)]$. *E. Grosswald* (Princeton, N.J.)

4190:

Green, J. A. A lifting theorem for modular representations. Proc. Roy. Soc. London. Ser. A **252** (1959), 135–142.

This paper gives conditions in terms of certain homological invariants for a modular representation of a group to be lifted to (induced by) an integral representation over the corresponding ring $\mathfrak{o}$ of $\mathfrak{p}$-adic integers. It is not assumed here that the group is finite. Let k be the residue-class field $\mathfrak{o}/\mathfrak{p}$. If Λ is the representation space for a modular representation of G over k, then Λ acquires a natural structure as a (k, G)-module for which the groups $\mathrm{ext}^s(\Lambda,\Lambda)$ can be constructed in the usual sense of homological algebra. Theorem: Let G be any group and ρ a modular representation of G on Λ. Then ρ can be lifted to an integral representation of G over $\mathfrak{o}$ if $\mathrm{ext}^2(\Lambda,\Lambda)=0$. If ρ_H is the restriction of ρ to a subgroup $H\subset G$, and ρ_H can be lifted, then similar criteria are given for extending the lifting to ρ. *W. E. Jenner* (Lewisburg, Pa.)

4191:

Burgess, D. C. J. Generalized intervals in partially ordered groups. Proc. Cambridge Philos. Soc. **55** (1959), 165–171.

If two elements a, b of a partially ordered set P have both a common upper bound and a common lower bound (notation: $a\sim b$), then the set-theoretic intersection of all intervals containing both a and b is called the D-interval $\langle a,b\rangle=\langle b,a\rangle$. It is shown that D-intervals have many properties in partially ordered groups G which are reminiscent of those of lattice-ordered groups, e.g., if $a\sim b$, then (i) $\langle -a,-b\rangle=-\langle a,b\rangle$; (ii) $\langle x+a+y, x+b+y\rangle = x+\langle a,b\rangle+y$ for all $x, y\in G$; (iii) $a-\langle a,b\rangle+b=\langle a,b\rangle$; etc. Calling P D-distributive if $x\sim a\sim y$ and $\langle x,a\rangle=\langle y,a\rangle$ imply $x=y$, the author shows that an archimedean partially ordered group is D-distributive.

L. Fuchs (Budapest)

4192a:

Saitô, Tôru. Homomorphisms of left simple semigroup onto a group. Proc. Japan Acad. **34** (1958), 664–667.

4192b:

Saitô, Tôru. Supplement of "Homomorphisms of a left simple semigroup onto a group". Proc. Japan Acad. **35** (1959), 114.

Given a homomorphism φ of a semigroup S onto a group, the inverse image under φ of 1 is a subsemigroup of S, the kernel of the homomorphism. When S is a sesquilateral division semigroup [P. M. Cohn, J. London Math. Soc. **31** (1956), 181–191; MR **18**, 14] it was shown by the reviewer [Proc. London Math. Soc. (3) **8** (1958), 466–480; MR **20** #5245] that the homomorphism is completely determined by the kernel K: the inverse images of the group elements are just the right cosets Ka; moreover the kernels themselves were characterized. The author obtains a similar characterization of the kernels in a semigroup S which is left simple, i.e. where $a=xb$ has a solution x in S for all $a, b\in S$. If I is the set of elements $x\in S$ such that $xa=a$ for some $a\in S$, then the core J of S is defined as the subsemigroup generated by all solutions of equations $ax=b$ ($a, b\in I$). Now the kernels in a left simple semigroup S are precisely the subsemigroups K such that (i) $xK\subseteq Kx$ for all $x\in S$, (ii) with $a, b\in K$ any solution of $ax=b$ belongs to K, (iii) $K\supseteq J$.—In the supplement it is

shown that the condition (iii) just given may be replaced by the condition that K be non-empty. {The core of a one-sided simple semigroup has also been considered by L. M. Gluskin, Dokl. Akad. Nauk SSSR **102** (1955), 673–676 [MR **17**, 237].} *P. M. Cohn* (Manchester)

4193:

Lefebvre, Pierre. Sur les sous-demi-groupes nets d'un côté, minimaux, d'un demi-groupe. C. R. Acad. Sci. Paris **248** (1959), 173–175.

Un complexe H d'un demi-groupe D est net à droite si $aD \cap H \neq \emptyset$ pour tout $a \in D$. Les résultats principaux: (1) Les sous-demi-groupes nets à droite minimaux sont aussi des complexes nets à droite minimaux. (2) Si S est un sous-demi-groupe de D net à droite minimal on a $sS = \{s\}$ pour tout $s \in S$. (3) Si D contient à la fois des complexes nets à droite minimaux et des complexes nets à gauche minimaux il suit de (2) que S est l'ensemble des idempotents de l'idéal à gauche le contenant. (4) D contient des sous-demi-groupes nets d'un côté minimaux si et seulement si D contient à la fois des complexes nets à droite minimaux et des complexes nets à gauche minimaux.

Št. Schwarz (Bratislava)

4194:

Lefebvre, Pierre. Sur les groupes homomorphes à un demi-groupe; demi-groupes admettant un groupe homomorphe maximum. C. R. Acad. Sci. Paris **248** (1959), 2277–2279.

Un demi-groupe D et un idéal (bilatère) de D admettent les mêmes groupes homomorphes. Quelques conséquences de ce théorème sont données. En particulier, si D possède un idéal (bilatère) complètement simple sans zéro, il admet un plus grand groupe homomorphe à D (au sens d'une relation d'ordre naturel), qui est le même que le groupe maximum correspondant à cet idéal.

Št. Schwarz (Bratislava)

4195:

Hennequin, Paul-Louis. Trajectoires définies par un semi-groupe d'applications d'un ensemble dans lui-même. C. R. Acad. Sci. Paris **248** (1959), 3108–3110.

Der Verfasser betrachtet einige von der Wahl einer Topologie unabhängige Eigenschaften einer Halbgruppe $(T_t)_{t\geq 0}$ von Abbildungen einer Menge E in sich. Bei beliebigem $x \in E$ bedeute A_x die Menge aller $T_t x$ mit $t \geq 0$. Durch $A_x \cap A_y \neq \emptyset$ wird eine Äquivalenzrelation $x \sim y$ definiert, vermöge derer E in Klassen B_ι zerfällt. Es sei $C_\iota = \bigcap \{A_x : x \in B_\iota\}$. Die Menge $\mathcal{T}(x) = \{t : t \geq 0, T_t x = x\}$ hängt dann innerhalb von C_ι nicht von x ab. Setzt man $\tau = \inf\{\mathcal{T}(x) \cap]0, +\infty[\}$, wenn $C_\iota \neq \emptyset$, so ist entweder $\tau > 0$, $\mathcal{T}(x) = \{n\tau : n \geq 0, n \text{ ganz}\}$ und C_ι hat die Mächtigkeit des Kontinuums, oder $\tau = 0$ und C_ι ist einelementig oder unendlich. Auf diese und ähnliche Feststellungen gestützt konstruiert der Verfasser dann unter gewissen Mächtigkeitsannahmen Halbgruppen $(T_t)_{t\geq 0}$ zu einer gegebenen Zerlegung von E in Klassen B_ι und gegebenen Mengen C_ι mit $C_\iota \subseteq B_\iota$ und gibt schließlich acht Beispiele an. *K. Krickeberg* (Aarhus)

4196:

Chaudhuri, Niranjan Prasad. Sur les demi-groupes inversifs et les demi-groupes réunions de groupes. C. R. Acad. Sci. Paris **248** (1959), 3262–3263.

A semigroup S is regular [inversif] if $x \in xSx$ for all x in S. If $x = xx'x$ then x' is a (quasi) inverse of x. The author shows that if S is regular and left cancellative then a subset H of S is right perfect [P. Dubreil, Mém. Acad. Sci. Inst. France (2) **63** (1941); MR 8, 15] if and only if when h_1, h_2, h_3 belong to H then $h_1 h_2' h_3 \in H$ for any inverse h_2' of h_2. *G. B. Preston* (Shrivenham)

4197:

Gluskin, L. M. Ideals of semigroups of transformations. Mat. Sb. (N.S.) **47 (89)** (1959), 111–130. (Russian)

Lyapin [Dokl. Akad. Nauk SSSR **88** (1953), 13–15; MR **15**, 395] has considered the structure of the semigroup $V(\Omega)$ of all one-to-one partial transformations of a set into itself, and the set $S(\Omega)$ of all mappings of a set Ω into itself. The principal tool was the concept of a densely imbedded ideal.

The author introduces the concept of a d-ideal: An ideal A of a semigroup $\bar{A}$ is called a d-ideal if it satisfies: (a) All non-trivial homomorphisms of the semigroup $\bar{A}$ induce non-trivial homomorphisms of the semigroup A; (b) If W' is any subsemigroup of $\bar{A}$ containing A and if some semigroup S contains W' with A as an ideal of S, then there exists a homomorphism of S to $\bar{A}$ inducing the identity isomorphism on W'. The relation between this concept and the previous work of Lyapin is contained in the following theorem: Let A be a d-ideal of a semigroup $\bar{A}$. In order that a semigroup D be isomorphic to the semigroup $\bar{A}$, it is necessary and sufficient that D contain a densely imbedded ideal isomorphic to A.

The author carries out an investigation of the semigroup $W(S)$ of all partial transformations of a set Ω into itself and extends the results of Lyapin on $V(\Omega)$ in terms of transitive subsemigroups and d-ideals.

L. J. Paige (Los Angeles, Calif.)

4198:

Hosszú, Miklós. A theorem of Belousov and some of its applications. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **9** (1959), 51–56. (Hungarian)

V. D. Belousov stated in one of his lectures [see Uspehi Mat. Nauk (N.S.) **13** (1958), no. 3 (81), 243] the following result for which a simple proof is given now: If F, G, H, K are binary operations on the same set Q such that each of them defines a quasigroup on Q and $F[G(x, y), z] = H[x, K(y, z)]$ for all $x, y, z \in Q$, then there exists a group (on Q) isotopic to all four quasigroups. Among the applications the author proves: if a quasigroup Q satisfies the property "$xy = uv$ if and only if $xu = yv$", then it is isotopic to an abelian group (written additively), and $xy = \alpha(x - y)$ where α is an invertible transformation of Q; if it satisfies "$xy = uv$ if and only if $xv = uy$", then it is isotopic to an elementary (abelian) 2-group.

L. Fuchs (Budapest)

TOPOLOGICAL GROUPS AND LIE THEORY

See also 4218, 4219, 4430, 4450, 4455.

4199:

Hulanicki, A. On the topological structure of 0-dimensional topological groups. Fund. Math. **46** (1959), 317–320.

The author proves that the underlying space of an infinite 0-dimensional compact group is homeomorphic to some (generalized) Cantor set (topological product of a suitable number of pairs of isolated points).

J. de Groot (Lafayette, Ind.)

4200:

Iwahori, Nagayoshi; and Iwahori, Nobuko. An extension of Tannaka duality theorem for homogeneous spaces. Sci. Papers Coll. Gen. Ed. Univ. Tokyo **8** (1958), 115–121.

The authors consider an extension of Tannaka's duality theorem to factor spaces of compact groups.

F. I. Mautner (Paris)

4201:

Chen, Kuo-Tsai. On the composition functions of nilpotent Lie groups. Proc. Amer. Math. Soc. **8** (1957), 1158–1159.

Let G be a Lie group and $x \to (x_1, \cdots, x_n)$ a system of canonical coordinates on G. If G is nilpotent, each $(xy)_i$ is a polynomial in $x_1, \cdots, x_n, y_1, \cdots, y_n$ due to the Baker-Hausdorff formula. The author gives a proof of the converse statement. *S. Helgason* (New York, N.Y.)

4202:

Harish-Chandra. Automorphic forms on a semisimple Lie group. Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 570–573.

Let G be a connected semisimple Lie group with finite center and K a maximal compact subgroup of G. Let Γ be a discrete subgroup of G and σ and μ unitary representations of K and Γ respectively on a finite-dimensional Hilbert space U. $\sigma(K)$ acts on U on the left and $\mu(\Gamma)$ acts on U on the right. Let $\mathfrak{Z}$ be the (commutative) algebra of differential operators on G which are invariant under all left and right translations of G. Fix a homomorphism χ of $\mathfrak{Z}$ into the field of complex numbers. A C^∞ function f from G to U is called an automorphic form of type (σ, μ, χ) if (1) $f(kx\gamma) = \sigma(k) f(x) \mu(\gamma)$ $(k \in K,\ x \in G, \gamma \in \Gamma)$ and (2) $zf = \chi(z) f$ $(z \in \mathfrak{Z})$. Such a function is always analytic. The author studies the space $\mathfrak{F}_0(\sigma, \mu, \chi)$ of automorphic forms on G which satisfy a certain growth condition at infinity. The main result is that $\dim \mathfrak{F}_0(\sigma, \mu, \chi) < \infty$ under a suitable condition on Γ (too complicated to describe here). This condition is stronger than finiteness of the volume of G/Γ. However it is fulfilled, for example, when $\Gamma = \mathrm{Sp}(n, Z)$ (Siegel's modular group) and $G = \mathrm{Sp}(n, R)$. The result is applied to the decomposition of the representation λ of G on $L^2(G/\Gamma)$ induced by the action of G on G/Γ. It is shown that each discrete irreducible component of λ occurs only a finite number of times in λ.

S. Helgason (New York, N.Y.)

MISCELLANEOUS TOPOLOGICAL ALGEBRA

See also 4314.

4203:

Clifford, A. H. Connected ordered topological semigroups with idempotent endpoints. II. Trans. Amer. Math. Soc. **91** (1959), 193–208.

In this paper the author completes the study of the semigroups described in the title. A complete classification of all such semigroups is given, which however is too long to state here. Part I [same Trans. **88** (1958), 80–98; MR **20** #1727] dealt with the case when the semigroup S has a zero. In the general case the Rees quotient $T = S/K$ (K is the minimal ideal) has a zero, and so the results of part I may be applied to T. {In a note to the reviewer the author points out that there is an omission in the text. In the converse half of theorem 8 the mapping θ should be assumed to satisfy $\theta(0) = \omega$.}

A. Shields (New York, N.Y.)

4204:

Mostert, Paul S.; and Shields, Allen L. Semigroups with identity on a manifold. Trans. Amer. Math. Soc. **91** (1959), 380–389.

The following general question arises quite naturally in the study of topological semigroups whose space is a manifold: if a manifold admits a continuous associative multiplication with identity and no other idempotents is it a group? In the first portion of this paper the authors give an affirmative answer in the cases that the manifold is the line or the plane.

Although the general question is still open, the authors prove the following theorem which applies to the general situation: If a manifold M is a semigroup with identity then there exists a maximal connected open subgroup of the identity.

After consideration of the cases of the line and the plane, the remainder of the paper is devoted to the following special problem: if the closed right half-plane is a semigroup such that the open right half-plane is a group G, what possibilities are there for multiplication on the y-axis? In the case G is isomorphic to the 2-dimensional vector group, the authors prove there are exactly four such possibilities. An example of each is given. If, however, G is isomorphic to the non-Abelian group of affine transformations of the line: $y = ax + b$, $a > 0$, two possibilities are given but the question is left open as to whether these are all. *A. Lester* (New Orleans, La.)

4205:

Dyer, Eldon; and Shields, Allen. Connectivity of topological lattices. Pacific J. Math. **9** (1959), 443–448.

A space X is acyclic if the reduced cohomology ring $H^*(X)$ is trivial; X is clc if for each $x \in X$ and each closed neighborhood U of x there is a closed neighborhood V of x such that $V \subset U$ and the homomorphism $H^*(U) \to H^*(V)$ induced by the inclusion is trivial. The authors prove that every compact connected topological lattice is acyclic and clc and hence, if the lattice is finite-dimensional, it has the fixed point property. If the underlying topology is metrizable then a compact connected topological lattice is contractible and locally contractible and hence, if it is finite-dimensional, it is an absolute retract. It is also shown that a compact finite-dimensional distributive topological lattice has a neighborhood basis of convex sublattices.

L. W. Anderson (Eugene, Ore.)

4206:

Anderson, Lee W. On the breadth and co-dimension of a topological lattice. Pacific J. Math. **9** (1959), 327–333.

In what follows the breadth of a lattice L is the smallest

integer n such that any finite subset F of L has a subset F' of at most n elements, such that $\inf F = \inf F'$. Codim L denotes the cohomology dimension of L. The center of a lattice with 0, 1 is the set of all x other than 0 and 1 such that for some $y \in L$, $x \wedge y = 0$ and $x \vee y = 1$.

The following theorems are established, giving partial solutions to conjectures of Dyer and Shields [#4205] and Wallace [Proc. Amer. Math. Soc. **7** (1958), 250–252; MR **20** #825]. (1) If L is a locally compact distributive topological lattice and if each pair of comparable points is contained in a closed connected chain, then breadth $L \leqq$ codim L. (2) If L is a compact connected distributive topological lattice and if codim $L \leqq n$ then the center of L contains at most $2^n - 2$ elements.

R. J. Koch (Baton Rouge, La.)

FUNCTIONS OF REAL VARIABLES

See also 4401, 4411, 4435.

4207:

★**Hardy, G. H. A course of pure mathematics.** 10th ed. Cambridge University Press, New York, 1958. xii + 509 pp. Paperbound: $3.75; 22s. 6d.

A reprinting, labeled "Student's edition", of the 10th ed. [1952; MR **14**, 145].

4208:

★**Pierpont, James. Lectures on the theory of functions of real variables.** 2 Vols. Dover Publications, Inc., New York, 1959. xii + 560 + xiii + 645 pp. Paperbound: $2.45 per vol.

Unaltered republication of the last edition [Ginn, Boston-New York, 1905, 1912].

4209:

★**Sikorski, Roman. Funkcje rzeczywiste.** [Real functions.] Vol. 2. Monografie Matematyczne. Tom 37. Państwowe Wydawnictwo Naukowe, Warsaw, 1959. 262 pp. zł. 35.00.

This completes the author's treatise on real variables [Vol. 1 (1958) was reviewed in MR **19**, 945]. There are four chapters: 12. Function spaces (linear spaces, L^p and other B-spaces, linear and bilinear operators and functionals, convolutions, function rings, distributions); 13. Hilbert space. Orthogonal series (inner product spaces, orthogonal systems, convergence a.e.); 14. Fourier series (periodic functions, elementary properties, divergence theorems, Fejer's theory, absolute convergence, L^2-theory, periodic distributions); 15. Fourier integrals (Fourier and Fourier-Stieltjes transforms, convergence criteria, L^2-theory). A 7-page bibliography and three indexes (names, symbols, general) are included. There are hundreds of problems, many covering certain important material not covered in the text. Due to the wealth of material considered, the author is forced to be selective and certain important topics are omitted (this is carefully explained in several places). {Some remarks: no knowledge of algebra is assumed and this seems to hamper the development in a few spots; Hilbert spaces are defined to be separable; the terms eigenvalue, spectrum, etc. are not even defined. The author says: "Banach is the creator of functional analysis"—surely Hilbert, Volterra, Frechet, etc., deserve some credit.} Although this volume seems to be somewhat inferior to the first one, the two form an excellent treatise and the Polish students and mathematicians are to be congratulated. *Z. A. Melzak* (Murray Hill, N.J.)

4210:

Obrechkoff, Nikola. Sur la convergence du procédé itératif dans l'analyse. I. C. R. Acad. Bulgare Sci. **12** (1959), 97–99. (Russian summary)

Let ω be a nonconstant map of the closed interval $[a, b]$ into itself. Let $\omega_1 = \omega$ and for $n = 2, 3, \cdots$, let ω_n be the composition of ω with ω_{n-1}. Assume (*) $|\omega(x) - \omega(y)| < |x - y|$ for $x, y \in [a, b]$, $x \neq y$. Then, for any $a_0 \in [a, b]$ the sequence $\omega_n(a_0)$ converges to the unique solution of $\omega(x) = x$. The condition (*) replaces the stronger known sufficient condition $|\omega(x) - \omega(y)| < q|x - y|$, $q < 1$. A similar theorem holds for the open interval (a, b) provided ω has a fixed point in (a, b). There is an application to the functional equation $y = f(x, y)$.

R. R. Goldberg (Evanston, Ill.)

4211:

Hartman, Philip. Unrestricted n-parameter families. Rend. Circ. Mat. Palermo (2) **7** (1958), 123–142.

A family F of continuous real-valued functions defined on an interval I is called an n-parameter family if there exists one member of the family taking on prescribed values at n prescribed distinct points x_i. If the members of F are of class C^{n-1}, then F is called an unrestricted n-parameter family if the above is true even when the x_i are allowed to coincide in any way; it being understood that when k of the x_i coincide, then the value of the function and of its first $k-1$ derivatives are prescribed at the point. The principal result of the paper is that on an open interval I the n-parameter family F is an unrestricted n-parameter family if and only if the above condition holds when all n of the points coincide; that is, if and only if there is exactly one solution from F of the initial value problem in which the values of f and the first $n-1$ derivatives are prescribed. In addition, there are theorems on differentiability and smoothness of functions convex with respect to unrestricted n-parameter families, and applications to differential inequalities.

J. W. Green (Los Angeles, Calif.)

4212:

Lahiri, B. K. A note on derivatives. Bull. Calcutta Math. Soc. **50** (1958), 68–70.

The author establishes slight extensions of a theorem due to J. A. Clarkson [Bull. Amer. Math. Soc. **53** (1947), 124–125; MR **8**, 451]. He alleges that the derivatives of functions of a certain class are not Riemann-integrable, but the details of his proof are not clear to the reviewer.

U. S. Haslam-Jones (Oxford)

4213:

Bellman, Richard. On a general method in the theory of inequalities. Rev. Ci. Lima **59** (1957), 21–33.

This paper discusses Minkowski's inequality, some inequalities of Beckenbach [Amer. Math. Monthly **57** (1950), 1–6; MR **11**, 422], Bergström [Den 11te Skandinaviske Matematikerkongress, Trondheim, 1949, pp. 264–

207; MR 14, 716], Ostrowski and Taussky [Nederl. Akad. Wetensch. Proc. Ser. A 54 (1951), 383–385; MR 13, 900] and the reviewer [Proc. Nat. Acad. Sci. U.S.A. 36 (1950), 31–35; MR 11, 526]. The purpose is to point out certain common features in the proofs of these inequalities.

Ky Fan (Notre Dame, Ind.)

4214:

Teicher, Henry. A question concerning positive type polynomials. Proc. Amer. Math. Soc. 10 (1959), 482–489.

The author presents solutions to certain problems on positive type polynomials recently proposed by P. Slepian [Bull. Amer. Math. Soc. 64 (1959), 59]. The essential question, which is answered in this paper, is as follows: given a positive integer n, what is the least ε such that there exist positive numbers $b_1, b_2, \cdots, b_n$ with the property that the polynomial $(x^2-x+\varepsilon)\prod_{i=1}^{n}(x+b_i)$ has positive coefficients? In the concluding section the author returns to the topic of an earlier paper, which deals with the factorization of positive type polynomials [Proc. Amer. Math. Soc. 6 (1955), 195–201; MR 16, 1019].

W. Ledermann (Manchester)

4215:

Melzak, Z. A. Existence of certain analytic homeomorphisms. Canad. Math. Bull. 2 (1959), 71–75.

Let $\{X_i\}$ and $\{Y_i\}$ each be sequences of pairwise-disjoint, countable, dense subsets of the unit interval. Generalizing a theorem of P. Franklin's [Trans. Amer. Math. Soc. 27 (1925), 91–100], the author shows that in the class G of order-preserving homeomorphisms of the unit interval [0, 1] onto itself there is an analytic function f such that for each $i=1, 2, \cdots$ the function f maps the set X_i onto the set Y_i. Franklin's methods then yield corollaries assuring, e.g., that any function belonging to class G can be uniformly approximated by such functions f.

T. A. Botts (Charlottesville, Va.)

MEASURE AND INTEGRATION

See also 4231, 4460, 4481.

4216:

★Strydom, B. C. Abstract Riemann integration. Getal en figuur, 10. Van Gorcum & Co. N. V., Assen, 1959. v+46 pp. Hfl. 5.25.

The objective of this dissertation is to develop a theory of finitely additive (s-additive) integration after the manner used by A. C. Zaanen in his *Introduction to the theory of Integration* [North-Holland Publishing Co., Amsterdam, 1958; MR 20 #3950] for the sequentially additive (σ-additive) case. This involves the extension of an s-additive measure function on a semi-ring of subsets of a fundamental set to a similar measure function on a ring of sets. Since a positive linear functional I on a vector lattice L of bounded functions on a set X defines an s-additive measure function on the ordinate set L^+ of positive functions of L, the extension of this measure function gives rise to an s-additive integral I on an extended class of functions as well as a class of I-measurable subsets of X. The Riemann integral enters as usual via the linear extension of this class of (I-integrable) characteristic functions on I-measurable sets. Further results parallel those of L. H. Loomis [Amer. J. Math. 76 (1954), 168–182; MR 15, 631] and H. Bauer [Math. Z. 65 (1956), 448–482; MR 18, 645].

T. H. Hildebrandt (Ann Arbor, Mich.)

4217:

Michael, J. H.; and Rennie, B. C. Measurability of functions of two variables. J. Austral. Math. Soc. 1 (1959/61), part 1, 21–26.

L'auteur étudie des problèmes intimement liés au théorème de Fubini. Le résultat le plus important est que si f est une fonction définie sur un domaine du plan mesurable au sens de Lebesgue, continue par rapport à l'une des variables et mesurable par rapport à l'autre, alors elle est mesurable dans le plan.

A. Fuchs (Strasbourg)

4218:

Urbanik, K. On the isomorphism of Haar measures. Fund. Math. 46 (1959), 277–284.

Let G be an infinite group and T_i ($i=1, 2, 3$) a topology under which G becomes a compact topological group G_i. Let B_i denote the σ-field of all Baire subsets of G_i and μ_i the Haar measure defined on B_i with $\mu_i G_i = 1$. Let A_i be the Boolean σ-algebra derived from B_i reducing modulo the σ-ideal consisting of the elements of B_i with measure zero. The Haar measure μ_i determines a measure λ_i in A_i in a natural way. μ_1 and μ_2 are said to be almost isomorphic if the corresponding λ_1 and λ_2 are isomorphic under a measure-preserving isomorphism between A_1 and A_2. Two subfields of a σ-field with measure μ are said to be independent if every two elements, one out of each subfield, are (stochastically) independent. Answering a problem raised by S. Hartman the main results are the following. (i) Given G and T_1 and T_2, there exists a compact topology T_3 on G such that B_3 contains two independent σ-subfields C_1 and C_2 for which $\mu_3|C_j$ and μ_j ($j=1, 2$) are almost isomorphic. (ii) If $(2^{\aleph_\alpha}=2^{\aleph_\beta}) \Rightarrow (\aleph_\alpha=\aleph_\beta)$, then for all compact topologies T on G the corresponding Haar measures on the σ-fields B are almost isomorphic.

J. de Groot (Lafayette, Ind.)

4219:

Mibu, Yoshimichi. On measures invariant under given homeomorphism group of a uniform space (A generalization of Haar measure). J. Math. Soc. Japan 10 (1958), 405–429.

In the ordinary development of Haar measure, we work with a locally compact group. In this paper, it is shown that we need only consider a locally compact topological space over which a group of homeomorphisms is defined, providing two conditions hold. It is then shown that the ordinary development of Haar measure is still valid.

Theorem: If Ω is a locally compact and σ-compact space whose topology is defined by a uniform system of symmetric neighborhoods $\{V_\alpha, \alpha\in\Theta\}$, let G be a group of homeomorphisms of Ω satisfying (c_I) $\sigma V_\alpha(p)=V_\alpha(\sigma p)$, all $\sigma\in G$, $p\in\Omega$, $\alpha\in\Theta$, and (c_{II}) for any p, $q\in\Omega$, there is a $\sigma\in G$ such that $\sigma(p)=q$. Then there exists a unique (up to a multiplicative constant) Baire measure which is regular, which measures open sets >0, compact sets less than ∞, and is invariant under the homeomorphisms in G.

The author also constructs the compact-open topology in G and proves some theorems about this as a topological

group. There are other theorems which extend the similarity to Haar measure.

The author notes that his results are similar to certain work of A. Weil in his book on integration in groups: "Most of the results in the present paper was (sic) announced earlier in a note [Proc. Japan Acad. **22** (1946), no. 8, 246–248; MR **12**, 810]. However, the publication of the details has been delayed owing to the author's health. Recently, Prof. K. Yosida communicated to the author that the results are closely related to the 'Mesure dans les espaces homogènes' in A. Weil's book, to which the author had not access when the note was published. In fact, if the space Ω is compact, then it will be proved without difficulty that our G-invariant measure can be introduced in Ω by making use of Weil's results. But it seems to the author that such deduction will not be possible for non-compact spaces."

A. Beck (Madison, Wis.)

4220:

Maharam, Dorothy. On a theorem of von Neumann. Proc. Amer. Math. Soc. **9** (1958), 987–994.

Es sei (S, F, m) ein Maßraum, wobei das Maß m nicht negativ, komplett, σ-additiv, σ-endlich mit $m(S)>0$ auf F vorausgesetzt ist. Es sei ferner N das Ideal der m-Nullmengen. Dann kann man jeder Restklasse $x \in F/N$ eine Teilmenge $R(x) \subseteq S$ derart zuordnen, daß $R(x)$ der Restklasse x angehört und ferner $R(0)=\emptyset$, $R(-x)=S-R(x)$, $R(x \wedge y)=R(x) \cap R(y)$,—also auch $R(x \vee y)=R(x) \cup R(y)$—gilt; d.h. es existiert stets ein endlicher Isomorphismus R der Booleschen Algebra F/N auf einer Booleschen Unteralgebra der Mengenalgebra F. Dieser Satz wurde zuerst von J. von Neumann [J. Reine Angew. Math. **165** (1931), 109–115] für den Fall des Lebesgueschen Maßes auf der Zahlengerade bewiesen. Später (1942) hat er mündlich Kakutani und der Verfasserin einen allgemeineren Beweis mitgeteilt. Verfasserin gibt nun einen neuen allgemeinen Beweis dieses Satzes. Die Boolesche Maß-Algebra F/N lässt sich nämlich, wie Verfasserin gezeigt hat [Proc. Nat. Acad. Sci. **28** (1942), 108–111; MR **4**, 12], in homogene Boolesche Maßalgebren zerlegen. Sie zeigt nun, daß der Satz für homogene Maßalgebren gilt. Daraus folgt leicht die allgemeine Gültigkeit des Satzes.

D. A. Kappos (Athens)

4221:

Maharam, Dorothy. On two theorems of Jessen. Proc. Amer. Math. Soc. **9** (1958), 995–999.

Verfasserin beweist folgende Verallgemeinerung eines Satzes von Jessen [Acta Math. **63** (1934), 249–323]: Es sei α^* eine beliebige ordinale Limeszahl und für jede Zahl $\alpha<\alpha^*$ X_α ein Maßraum mit einem nicht negativen, σ-additiven, kompletten und normierten Maß. Es bedeute $X=\prod\{X_\alpha, \alpha<\alpha^*\}$ den Produktmaßraum mit Komponenten X_α, $\alpha<\alpha^*$. Man betrachte ferner die Produktmaßräume $Y_\gamma=\prod\{X_\alpha, \alpha<\gamma\}$, $Z_\gamma=\prod\{X_\alpha, \gamma \leqq \alpha<\alpha^*\}$ für jede Zahl γ mit $1<\gamma<\alpha^*$. Für einen Punkt $x=\{x_\alpha, \alpha<\alpha^*\} \in X$ setze man $y_\gamma=\{x_\alpha, \alpha<\gamma\} \in Y_\gamma$ und $z_\gamma=\{x_\alpha, \gamma \leqq \alpha<\alpha^*\} \in Z_\gamma$. Dann kann man x als ein Paar (y_γ, z_γ) und demgemäss den Maßraum X als identisch zu dem Produktmaßraum $Y_\gamma \times Z_\gamma$ für jede Zahl γ betrachten. Es sei nun f eine reellwertige Funktion auf X definiert, die summierbar ist, also $\int_X |f|dx<+\infty$ gilt. Dann ist $f(x) \equiv f(y_\gamma, z_\gamma)$, als Funktion nur von y_γ bzw. nur von z_γ betrachtet, auch summierbar auf Y_γ bzw. auf Z_γ. Setzt man nun $f_\gamma(x) \equiv f_\gamma(y_\gamma, z_\gamma)=\int_{z_\gamma} f(y_\gamma, z_\gamma)dz_\gamma$ bzw. $f^\gamma(x) \equiv f^\gamma(y_\gamma, z_\gamma)=\int_{Y_\gamma} f(y_\gamma, z_\gamma)dy_\gamma$ so gilt $\lim_{\gamma \to \alpha^*} f_\gamma(x)=f(x)$ bzw. $\lim f^\gamma(x)=\int_X f(x)dx$ für fast jeden Punkt $x \in X$. Für $\alpha^*=\omega_0$ (=die erste ordinale Limeszahl) hat man dann den Satz von Jessen.

D. A. Kappos (Athens)

4222:

Revuz, André. Seconde formule de la moyenne dans les espaces topologiques ordonnés. Arch. Math. **10** (1959), 153–154.

Dans cette note l'auteur énonce le théorème suivant [pour définitions et notations voir, par exemple, Ann. Inst. Fourier **6** (1955), 187–268; MR **19**, 536].

X est un espace topologique ordonné qui est: (i) de type $\tilde{\mathfrak{X}}$, (ii) K_σ, (iii) connexe. Sur X sont définies deux fonctions de répartition réelles U et V: U est totalement croissante et continue à droite; V est à variation bornée et continue à droite et la fonction duale V^* est continue. On suppose, en outre, que pour la mesure m_V associée à U, $m_V(X)$ est finie et que $\int_X U\,dV$ est finie. Il existe alors un élément ξ de X tel que $\int_X U\,dV=V^*(\xi)m_V(X)$.

A. T. Bharucha-Reid (Eugene, Ore.)

4223:

Koshi, Shōzō. On semi-continuity of functionals. II. Proc. Japan Acad. **35** (1959), 122–126.

The author applies the results of part I [same Proc. **34** (1958), 513–517; MR **20** #7206] to finitely additive functionals on Boolean algebras.

I. G. Amemiya (Tokyo)

FUNCTIONS OF A COMPLEX VARIABLE

See also 4260, 4313, 4319, 4320, 4343, 4397.

4224:

★**Pierpont, James. Functions of a complex variable.** Dover Publications, Inc., New York, 1959. xiv+583 pp. Paperbound: $2.45.

Unaltered republication of the 1st ed. [Ginn, Boston-New York, 1914].

4225:

★**Knopp, Konrad. Aufgabensammlung zur Funktionentheorie. II. Aufgaben zur höheren Funktionentheorie.** 5te Aufl. Sammlung Göschen Bd. 878. Walter de Gruyter & Co., Berlin, 1959. 151 pp. DM 3.60.

The fourth edition, with which the present one is essentially identical, was published in 1949 and translated into English in 1953 [see MR **14**, 1073]. A translation of vol. 1 was published by Dover [New York, 1948; MR **10**, 288].

4226:

Nisigaki, Hisami; and Takasu, Tsurusaburo. General three-dimensional complex function theory. II. Yokohama Math. J. **4** (1956), 147–226.

This consists of "§ 3, Theory of differentiation and differentiable functions" and follows the outline of the

table of contents given in I [same J. **3** (1955), 53–126; MR **18**, 725].

Let $f(\zeta)=u_1+E_2u_2+E_3u_3$ be a function of the 3-dimensional complex variable $\zeta=x_1+E_2x_2+E_3x_3$. If the differential quotient $\Delta f(\zeta)/\Delta\zeta$ exists independent of the direction of $\Delta\zeta$, it is defined to be the derivative $f'(\zeta)$. $f(\zeta)$ is called regular in a domain B if $f'(\zeta)$ exists throughout B. The authors determine a set of generalized Cauchy-Riemann differential equations and a set of generalized Laplace equations. Power series and analytic functions are defined. Extensions of mean value theorem, Rolle's theorem, etc., are derived.

J. A. Ward (Holloman Air Force Base, N.M.)

4227:

Nisigaki, Hisami; and Takasu, Tsurusaburo. General three-dimensional complex function theory. III. Yokohama Math. J. **5** (1957), 1–98.

This is the third installment (consisting of five sections) of the comprehensive treatment of the function theory for these 3-dimensional algebras and follows the table of contents given in I [see preceding review]. Section 4, Special functions, discusses generalizations of exponential functions, trigonometric functions, etc. Section 5, Integration theory, discusses line integrals, change of variables, Weierstrass mean value theorem, etc. Sections 6 and 7 are on polygenic functions. Extensions are given of Cauchy's integral theorem and Cauchy's integral formula. Section 8, Analytic functions, studies Taylor's series, residues, and treats special problems that arise with nilfactors.

J. A. Ward (Holloman Air Force Base, N.M.)

4228:

Passaquindici, Maria. Sulla stabilità dei polinomi e delle matrici. Ann. Scuola Norm. Sup. Pisa (3) **13** (1959), 77–88.

The first part of this study is concerned with an exposition of sufficient conditions for a real definite matrix of order $n>2$ to have all zeros with negative real parts [cf. S. Cherubino and M. Passaquindici, Ann. Scuola Norm. Sup. Pisa (3) **12** (1958), 31–53; MR **20** #3626]. These conditions are consequences of theorems of Lévy and Hadamard, Müller, Ostrowsky. By use of these results, sufficient conditions are prescribed for a polynomial to be a Hurwitz polynomial. In addition, sufficient conditions are given for the conservation of the stability of a polynomial (all zeros have negative real parts) when increments are given to the coefficients; also conditions for the conservation of the stability of the characteristic polynomial of a matrix when the elements of a row or column are given increments.

E. Frank (Chicago, Ill.)

4229:

Thron, W. J. Zwillingskonvergenzgebiete für Kettenbrüche $1+K(a_n/1)$, deren eines die Kreisscheibe $|a_{2n-1}|\leq\rho^2$ ist. Math. Z. **70** (1958/59), 310–344.

The twin convergence region problem considered here is that of finding a region G_ρ such that the condition $|a_{2n-1}|\leq\rho^2$, $|a_{2n}|\in G_\rho$, implies the convergence of the continued fraction $1+a_1/1+a_2/1+\cdots$. Singh and Thron [Proc. Amer. Math. Soc. **7** (1956), 277–282; MR **17**, 1076] proved that the condition (i) $|a_{2n-1}|\leq\rho^2$, $a_{2n}=c_{2n}^2$, $|c_{2n}\pm i|\geq\rho$, is sufficient for convergence of the continued fraction when $0<\rho<1$, and that the twin convergence regions thus obtained were best in a sense. In Part I of the present paper this theorem is re-proved and it is shown that the value of the continued fraction satisfies (ii) $|z-1|\leq\rho$. If the condition (iii) $|a_{2n}|\geq\rho^2-1+\varepsilon$, $0<\varepsilon<1$, is adjoined to condition (i), when $\rho=1$ the continued fraction converges to a value satisfying (ii), and when $\rho>1$ the sequence of even approximants converges to a value satisfying (ii).

Cowling, Leighton and Thron [Bull. Amer. Math. Soc. **50** (1944), 351–357; MR **6**, 47] showed that condition (iv) $|a_{2n-1}|\leq r-\varepsilon_1$, $|a_{2n}|\geq 2[r+\varepsilon_2-\cos\arg a_{2n}]$, $r>1$, $0<\varepsilon_i<r$ $(i=1,2)$, is sufficient for convergence of the continued fraction. The principal result of part II is that the continued fraction converges and its value satisfies $|z-1|\leq 1$ when condition (iv) holds with $\varepsilon_1=\varepsilon_2=0$. The result is best in a sense.

Results on uniform convergence are given in parts I and II. Proofs employ the method of nested circles.

W. T. Scott (Evanston, Ill.)

4230:

Kovlov, A. I. Distribution of singular points of functions represented by series of Dirichlet polynomials. Mat. Sb. N.S. **45 (87)** (1958), 489–510. (Russian)

This article is concerned with the location of singularities of the Hadamard product $\phi(z)=\sum a_nb_nz^n$ of two power series $f(z)=\sum a_nz^n$ and $\varphi(z)=\sum b_nz^n$ when $\varphi(z)$ has the unique singularity $z=1$. It also deals with the generalization to Dirichlet series of the Hadamard product by considering the relation between $\sum a_ne^{-\lambda_n t}$ and $\sum a_n\theta(n)e^{-\lambda_n t}$ with $\theta(z)$ an entire function of minimum type order unity. The function $\phi(z)$ is regarded as the result of an operator acting on $g(t)=f(e^t)$ of the form $A(g)=\sum_0^\infty c_kg^{(k)}(t)$. The solution of the equation $A(g)=\omega(t)$ is also considered. Operators of the form $P(z)=\sum a_jf(\lambda_jz)$ are discussed. The article is partly expository with references to the last 30 years.

An example of the results obtained is Theorem 2: If α is singular for the function $g(t)$ but not for $\sum_0^\infty c_ng^{(n)}(t)$ where $\sum_0^\infty c_nz^n$ is an entire function of minimum type order unity, then $g(t)=g_0(t)+F(t)$ where $g_0(t)$ is regular at α while $F(t)$ is singular at α and its domain of existence is convex.

A. J. Macintyre (Cincinnati, Ohio)

4231:

Ionescu Tulcea, Alexandra. Sur le prolongement analytique des séries aléatoires. C. R. Acad. Sci. Paris **248** (1959), 3396–3397.

Let τ be a measure-preserving mapping of the probability space $(X,\mathscr{B},\mu)$ into itself. Theorem 1: τ is ergodic $\Leftrightarrow$ for each non-null f in $L^1(X,\mathscr{B},\mu)$, the radius of convergence of $\sum_{n=0}^\infty f(\tau^n(x))z^n$ is 1, for a.e. x.

Other theorems assume ergodicity of τ or its powers, and conclude that, for various classes of functions f, the natural boundary of $\sum_{n=0}^\infty f(\tau^n(x))z^n$ is the unit circle, for a.e. x. Theorem 4: For a unitary U on infinite-dimensional Hilbert space $\mathscr{H}$, the following are equivalent. (i) $\{V^{-1}UV \mid V \text{ unitary}\}$ is strongly dense in all unitaries. (ii) The spectrum of U is the unit circle. (iii) The set of f in $\mathscr{H}$, for which $\sum_{n=0}^\infty (U^{n+1}f)z^n$ (as an analytic vector-valued function) has as its natural boundary the unit circle, is a dense G_δ. No proofs are given.

J. Feldman (Berkeley, Calif.)

4232:

Kahramaner, Suzan. Sur le comportement d'une représentation presque-conforme dans le voisinage d'un point singulier. Rev. Fac. Sci. Univ. Istanbul. Sér. A **22** (1957), 127–139. (Turkish summary)

"Dans le présent travail on donne la démonstration de deux théorèmes énoncés par R. Nevanlinna dans "Ueber fastkonforme Abbildungen" [Ann. Acad. Sci. Fenn. Ser. A. I. no. 251/7 (1958); MR **21** #1378] sur le comportement d'une représentation presque-conforme, c'est-à-dire différentiable et à différentielle continue, dans le voisinage d'un point singulier isolé. Le premier théorème exprime, sous certaines conditions, la valeur de la fonction en ce point et le second donne, sous d'autres, un développement asymptotique." (Author's summary)

H. L. Royden (Stanford, Calif.)

4233:

Shibata, Kêichi. On approximation of quasi-conformal mapping. Proc. Japan Acad. **35** (1959), 22–24.

Using results due to L. V. Ahlfors [Ann. Acad. Sci. Fenn. Ser. A. I. no. 206 (1955); MR **17**, 657] the author proves the following theorem: A given quasiconformal mapping $w=f(z)$ of $|z|<1$ onto $|w|<1$ which takes a set of given boundary points $\exp(i\theta_\nu)$ $(\nu=1, \cdots, k)$ into the given points $f(\exp(i\theta_\nu))=\exp(it_\nu)$, can be approximated uniformly on $|z|<1$ by quasiconformal mappings $w=f_n(z)$, $f_n(\exp(i\theta_\nu))=\exp(it_\nu)$ $(\nu=1, \cdots, k;\ n=1, 2, \cdots)$ which belong to the class C^1.

K. Strebel (Fribourg)

4234:

Mizumoto, Hisao. On conformal mapping of a multiply-connected domain onto a canonical covering surface. Kôdai Math. Sem. Rep. **10** (1958), 177–188.

The theorem that any domain of finite connectivity $N\geq 2$ with two assigned boundary components C_1 and C_2 can be mapped conformally onto a circular annulus with concentric circular slits, C_1 and C_2 corresponding to the inner and outer circle, and that the mapping is uniquely determined up to a transformation $w\to aw$, is generalized in the following sense: Given in the z-plane any domain B with boundary components $C_1, \cdots, C_N$ $(N\geq 2)$, none of which degenerates to a point, and to each C_j an integer ν_j, not all equal to zero and such that $\sum_{j=1}^{N}\nu_j=0$. There exists a function $w=\phi(z)$ analytic in B which maps the C_j onto circles or circular slits centered at the origin with the rotation numbers $\nu_j=(2\pi)^{-1}\int_{C_j} d\arg\phi(z)$. The image G of B is called a covering surface of annular type cut along concentric circular slits. ϕ can be normed by prescribing $\phi(z_0)=1$ for a given point $z_0\in B$, and it is then uniquely determined. A minimum property of these covering surfaces is given in terms of logarithmic area and a necessary and sufficient condition for a domain B of connectivity $2N$ that it can be mapped onto an N-times covered circular annulus $(\nu_j=\pm 1)$ in terms of the harmonic measures of the boundary components.

K. Strebel (Fribourg)

4235:

Jenkins, James A. A counter-example to a statement on conformal mapping of Riemann surfaces. Proc. Amer. Math. Soc. **10** (1959), 423–424.

The author points out that lemma 1 of the reviewer's paper in Trans. Amer. Math. Soc. **76** (1954), 14–25 [MR **15**, 519] is incorrect and gives a counter-example to show that the theorem in this paper is false unless some additional hypothesis is added such as the conclusion of lemma 1. The reviewer would like to point out that his theorem is a special case of and is superseded by the general coefficient theorem of Jenkins [ibid. **77** (1954), 262–280; MR **16**, 232]. *H. L. Royden* (Stanford, Calif.)

4236:

Andreyan Kazaku, Kabiriya [Andreian Cazacu, Cabiria]. Quasi-conformal mappings. Dokl. Akad. Nauk SSSR **126** (1959), 235–238. (Russian)

Let G denote a region of a Riemann surface and $\{K\}$ a family of rectifiable Jordan arcs and curves in G. A quasi-conformal transformation f defined in G is said to belong to the class O with respect to K provided that there exist a constant d and a univalent quasi-conformal transformation φ having the same family of infinitesimal ellipses as f and satisfying the inequality $l(K^*)\leq d^2 l(K)$ where K^* is the image of K with respect to φ and $l(K)$ and $l(K^*)$ are the extremal lengths of $\{K\}$ and $\{K^*\}$ respectively. Two subclasses of O are also defined. The author announces a number of generalizations of classical function-theoretic results for quasi-conformal transformations based on the use of the notion of the class O and its subclasses. These theorems include generalizations of the Bloch theorem, the Koebe $\frac{1}{4}$-theorem, and (for quasi-conformal transformations of finite valence) theorems of the Fatou, Lusin-Privaloff, and Riesz-Frostman-Nevanlinna types. Other results are announced, such as the invariance of parabolicity (resp. hyperbolicity) of boundary sets of a Riemann surface with respect to quasi-conformal maps of the class O (taken relative to certain families $\{K\}$).

M. H. Heins (Urbana, Ill.)

4237:

Constantinescu, Corneliu; et Cornea, Aurel. Comportement des transformations analytiques des surfaces de Riemann sur la frontière de Martin. C. R. Acad. Sci. Paris **249** (1959), 355–357.

Given a hyperbolic Riemann surface R, let Δ be its Martin boundary and Δ_1 the minimal part of Δ [Trans. Amer. Math. Soc. **49** (1941), 137–172; MR **2**, 292]. Consider an analytic transformation f of R into another Riemann surface R', and denote by f the analogue, in a specified sense, of Fatou's function. The following counterpart of the Fatou-Nevanlinna theorem is given: If R' is hyperbolic, then f is defined almost everywhere on Δ_1. This conclusion remains true if R' is parabolic but f is a Lindelöfian map in the sense of M. Heins [Ann. of Math. (2) **62** (1955), 418–446; MR **17**, 726]. Similar generalizations are announced of the Riesz-Lusin-Priwaloff theorem and of Löwner's lemma.

Applications to the classification theory of Riemann surfaces are then given: The class of hyperbolic Riemann surfaces that possess bounded minimal harmonic functions is discussed and connections are indicated with results of the authors [Nagoya Math. J. **13** (1958), 169–233; MR **20** #3273] and of Z. Kuramochi [Osaka Math. J. **5** (1953), 155–201; MR **15**, 518].

L. Sario (Los Angeles, Calif.)

4238:

Seibert, Peter. Über die Randstrukturen von Überlagerungsflächen. Math. Nachr. **19** (1958), 339–352.

The author considers a simply-connected Riemann surface covering the sphere and introduces an ideal boundary using the Mazurkiewicz metric. He shows that this boundary can be mapped continuously on a circle and that each continuum of the boundary is homeomorphic to the circle or to a closed circular arc. He considers the "direct boundary points" of Iversen and a more general class of "angularly accessible" boundary points and shows that if the surface is parabolic there are no direct boundary points and the angularly accessible points form a set of first Baire category.

H. Royden (Stanford, Calif.)

4239:

★Jørgensen, Vilhelm. On a minorant of the hyperbolic measure for a certain class of domains. Treizième congrès des mathématiciens scandinaves, tenu à Helsinki 18–23 août 1957, pp. 135–138. Mercators Tryckeri, Helsinki, 1958. 209 pp. (1 plate)

Let $f(z)$ be an analytic function regular in $\mathrm{Re}\, z>0$ which does not take values that cover any closed interval of length 2π on the imaginary axis. As a generalization of the Picard theorem transferred to the half plane, the author has proved that, as $z\to\infty$ along any Stolz path, $f(z)z^{-1}$ and $f'(z)$ tend to a real value c, and if $c=1$ there exists a minorant of $\mathrm{Re}\, f(z)$ for $\mathrm{Re}\, z>0$. In the present paper, it is shown that the best possible minorant is the same function, expressible in terms of the elliptic modular function, as in the Picard case where the missing values are $2p\pi i$ $(p=0, \pm 1, \cdots)$.

Y. Komatu (Tokyo)

4240:

Lehto, Olli. A generalization of Picard's theorem. Ark. Mat. **3** (1958), 495–500.

Let E be a closed set of the complex plane and let f be meromorphic in the complement of E with at least one singularity in E. E is called a Picard set if f cannot omit more than two values in the complement of E. The following sufficient conditions are given for E to be a Picard set. Let $E=\{a_1, a_2, \cdots, a_\nu, \cdots\}$, $a_\nu\to\infty$, and let $(\log|a_\nu|)^{2+\delta}=O(\log|a_{\nu+1}|)$, $\delta>0$; then E is a Picard set. For entire functions, E is called a Picard set if f cannot omit more than one finite value in the complement of E. Then E is a Picard set if $|a_\nu/a_{\nu+1}|=O(\nu^{-2})$. If the points of E lie on a ray, then $|a_{\nu+1}/a_\nu|\geq q>1$ implies that E is a Picard set for entire functions.

G. Springer (Lawrence, Kans.)

4241:

Hiong, King-lai. Sur les fonctions méromorphes en rapport avec leurs dérivées. Sci. Sinica **7** (1958), 661–685.

Complicated Nevanlinna type inequalities, several covering more than half a page, in which $T(r,f)$ is estimated in terms of $N(r,f)$, $N(r,f')$, $N(r,1/f)$, $N(r,1/f')$, $N(r,1/f^{(k)})$, $N(r,1/f^{(k+1)})$, p values $N(r,1/\{f-a_\nu\})$ and q values $N(r,1/\{f^{(k)}-b_\nu\})$.

J. Korevaar (Madison, Wis.)

4242:

Arens, Richard. The closed maximal ideals of algebras of functions holomorphic on a Riemann surface. Rend. Circ. Mat. Palermo (2) **7** (1958), 245–260.

Let M be a Riemann surface, G an open subset, and Z a closed subset of M containing G. Let $\mathrm{CH}(Z,G)=A$ be the class of complex-valued functions defined and continuous on Z and holomorphic in G. The author shows that each continuous homomorphism of A into the complex numbers is given by evaluation at a point of Z. If Z is compact, the requirement that the homomorphism be continuous may be dropped, and it can also be dropped if M can be realized as a finite-sheeted covering of the sphere (i.e. if there is a meromorphic function of bounded valence).

H. L. Royden (Stanford, Calif.)

4243a:

Hiong, King-lai. Sur le cycle de Montel-Miranda dans la théorie des familles normales. Sci. Sinica **7** (1958), 987–1000.

4243b:

Hiong, King-lai. Sur le cycle de Montel-Miranda dans la théorie des familles normales. Acta Math. Sinica **9** (1959), 76–86. (Chinese. French summary)

Let f be holomorphic and nowhere equal to 0 in the unit disc, and suppose that the kth derivative of f does not take the value 1. Improving on some of his earlier work [Ann. Sci. École Norm. Sup. (3) **72** (1955), 165–197; MR **17**, 600] the author proves the estimate

$$\log M(r) < \frac{1}{1-r}\{A\log^+|f(0)|+B\log^+|1/f(0)| +C\log[2/(1-r)]\}.$$

The second paper is a Chinese version of the first.

J. Korevaar (Madison, Wis.)

4244:

Hiong, King-lai. Sur la normalité des familles de fonctions holomorphes en rapport avec la théorie des défauts. Sci. Sinica **8** (1959), 1–18.

It is well known that a family of functions $f(z)$, holomorphic in $|z|<1$, is normal if the functions $f(z)$ are uniformly bounded in every disc $|z|\leq r<1$. Applying methods of value distribution theory, the author shows that, under certain conditions, uniform bounds for the modulus of $f(z)$ can be inferred from the behavior of $f(z)$ with respect to two complex values. He thus obtains various normality criterions which are considerably simpler than most of the previously derived conditions of this type. A typical result is the following. A family of functions $f(z)$ is normal if in $|z|<1$, $f(z)\neq\alpha$, a value $\beta\neq\alpha$ is deficient for every $f(z)$ with a deficiency greater than an absolute constant $\delta_0>0$, and $f(0)\neq\beta$. {The reviewer is not quite convinced of the necessity of the last condition.}

O. Lehto (Helsinki)

4245:

Väisälä, Jussi. On normal quasiconformal functions. Ann. Acad. Sci. Fenn. Ser. A. I. no. 266 (1959), 33 pp.

The author extends the recent results of Lehto and Virtanen [Acta Math. **97** (1957), 47–65; MR **19**, 403] on normal meromorphic functions to more general classes of functions, and, in particular, to quasiconformal functions.

A complex-valued function $w(z)$ is called by Lehto and Virtanen normal in a simply connected domain G if the family $\{w(S(z))\}$ is a normal family, where $S(z)$ is an arbitrary conformal mapping of G onto itself; for a multiply

connected domain, $w(z)$ is called normal if it is normal on the universal covering surface of G.

It is shown in the present paper that if $w(z)$ is continuous and normal in an elliptic or parabolic domain, then $w(z)$ is constant, and, for the non-trivial case of an hyperbolic domain, it is shown that $w(z)$ is normal in G if and only if there exists a non-decreasing function $\mu(r)$, defined for all $r \geq 0$ and with $\lim_{r\to 0} \mu(r) = 0$, such that $s(w(z_1), w(z_2)) \leq \mu(\sigma(z_1, z_2))$, $z_1, z_2 \in G$, where $s(a, b)$ and $\sigma(a, b)$ denote, respectively, the spherical and hyperbolic distances between a and b. If $w(z)$ is K-quasiconformal, $\mu(r)$ may be replaced here by $Mr^{1/K}$, where M is a finite constant; this bears a relation to a less general result of Marty [Ann. Fac. Sci. Toulouse (3) **23** (1931), 183–261]. The author then proves a decomposition theorem giving a relationship between normal meromorphic functions and normal quasiconformal functions: A quasiconformal function $w(z)$ is normal in G if and only if it admits a factoring of the form $w(z) = f(T(z))$, where $T(z)$ is a schlicht quasiconformal mapping of G onto a domain G' and $f(T)$ is a normal meromorphic function in G'. The paper concludes with various applications, in particular, with extensions to the case of quasiconformal functions of various results of Lehto and Virtanen concerning angular limits, normality constants, and the behaviour of such functions in the vicinity of an isolated essential singularity. *A. J. Lohwater* (Houston, Tex.)

4246:

Gehring, F. W.; and Lohwater, A. J. On the Lindelöf theorem. Math. Nachr. **19** (1958), 165–170.

Let D denote the unit circle $|z| < 1$, and let $P = e^{i\theta}$ be a point of its circumference Γ. If $f(z)$ is analytic and bounded in D and tends to a limit c as $z \to P$ along an arc γ in $D \cup \Gamma$, then by Lindelöf's Theorem, $f(z)$ has the sector limit c at P, i.e., $f(z) \to c$ uniformly as $z \to P$ in each sector $S_\Delta : |\arg(1 - ze^{-i\theta})| < \Delta$, $0 < \Delta < \pi/2$.

The authors prove the following generalization of Lindelöf's Theorem. Let $f(z) = u(z) + iv(z)$ be analytic and $|u(z)|$ bounded in D, and let $u(z) \to a$ as $z \to P$ along an arc α, $v(z) \to b$ as $z \to P$ along an arc β, where α and β lie in $D \cup \Gamma$. Then $f(z)$ has $a + ib$ as a sector limit at P. A counterexample shows that the condition $|u(z)| < M$ cannot be replaced by a pair of one-sided conditions such as $u(z) < M$, $v(z) < M$. However, if the arcs α and β lie in some sector S_Δ, then the condition $|u(z)|$ bounded can be replaced by the weaker requirement that $u(z)$ be one-sidedly bounded. *O. Lehto* (Helsinki)

4247:

Pólya, Georges; et Schiffer, Menahem. Sur la représentation conforme de l'extérieur d'une courbe fermée convexe. C. R. Acad. Sci. Paris **248** (1959), 2837–2839.

Let $\bar{r}$ be the outer conformal radius of a closed convex curve with the perimeter L. It is shown that the classical result $2\pi\bar{r} \leq L$ is supplemented by the converse estimate $L \leq 8\bar{r}$ in which the bound is attained when the curve degenerates to a segment. *Y. Komatu* (Tokyo)

4248:

Abe, Hitoshi. On univalent functions in an annulus. Math. Japon. **5** (1958/59), 25–28.

Let D denote the annulus $r < |z| < 1$, $r > 0$. The author applies the area principle of Meschkowski [Ann. Acad. Sci. Fenn. Ser. A. I. Math.-Phys. no. 117 (1952); MR **14**, 367] to obtain distortion theorems and coefficient estimates for functions f which are regular, except for a simple pole of residue 1 at $\zeta \in D$, and univalent in D. Let $\varphi_\theta(z, \zeta) = (z - \zeta)^{-1} + \sum_{n=-\infty}^{\infty} c_n z^n$, $c_0 = 0$, represent the function mapping D onto the parallel slit domain with slits of inclination θ, and let $N = \frac{1}{2}(\varphi_0 + \varphi_{\pi/2})$. If $f(z) - N(z, \zeta) = \sum_{n=-\infty}^{\infty} a_n z^n$, then $\sum_{n=-\infty}^{\infty} n(1 - r^{2n})|a_n|^2 \leq \pi K(\zeta, \zeta)$, where $K(z, \zeta)$ denotes the Bergman kernel function of D, and $|f'(z) - N'(z, \zeta)| \leq \pi K(\zeta, \zeta)^{1/2} K(z, z)^{1/2}$. Both of these estimates are sharp for $f = \varphi_\theta$ with $z = \zeta$ in the second estimate. He also obtains estimates for $|f(z) - a_0 - N(z, \zeta)|$ and for $|a_n|$, $n = \pm 1, \pm 2, \cdots$.

G. Springer (Lawrence, Kans.)

4249:

★Sario, Leo. On univalent functions. Treizième congrès des mathématiciens scandinaves, tenu à Helsinki 18–23 août 1957, pp. 202–208. Mercators Tryckeri, Helsinki, 1958. 209 pp. (1 plate)

Let W be a plane region of arbitrary connectivity containing the point $z = \infty$. For each φ in $0 \leq \varphi < \pi$, let $P_\varphi(z) = z + \sum_1^\infty a_n z^{-n}$ (about $z = \infty$) map W conformally onto a region whose boundary components are either points or linear slits with direction φ. Let γ_φ be the image of a component γ of the boundary of W. The author proves that γ_φ is either a point for every φ, or a slit for every φ, or otherwise a point for one and only one φ. Now let P_0 and P_1 denote the horizontal and vertical slit mappings and let $G = P_0 + P_1$ be the Grunsky function. It is next proved that G is univalent and maps each γ onto a convex curve or a point.

G. Springer (Lawrence, Kans.)

4250:

Oğuztöreli, M. Namık. Sur les fonctions à type borné. Rev. Fac. Sci. Univ. Istanbul. Sér. A **22** (1957), 141–149. (Turkish summary)

An analytic function is called a function of bounded type if it can be represented as the quotient of two bounded analytic functions. The author, using results from a previous paper [same Rev. **18** (1953), 384–419; MR **17**, 357], generalizes the Poisson-Stieltjes integral formula to multiply connected domains. This new integral formula is used to give a general representation of functions of bounded type in multiply connected domains which generalizes some of the works of Nevanlinna on "beschrankartige" functions [R. Nevanlinna, *Eindeutige analytische Funktionen*, Springer, Berlin, 1936; pp. 180–190]. The paper contains a number of misprints.

W. C. Royster (Lexington, Ky.)

4251:

Royster, W. C. Functions having positive real part in an ellipse. Proc. Amer. Math. Soc. **10** (1959), 266–269.

Let E be an ellipse with foci ± 1 and semi-axes $a > b$, $a > 1$. If $f(z)$ is regular in E, then $f(z)$ is representable in a series of Tchebycheff polynomials $T_n(z)$ with constant coefficients. The author proves that if $f(z)$ is regular and has a positive real part in E and

$$f(z) = 1 + \sum_{n=1}^{\infty} a_n T_n(z), \quad a_n = \alpha_n + i\beta_n,$$

then

$$\alpha_n^2(R^n+R^{-n})^2+\beta_n^2(R^n-R^{-n})^2 \leqq 16, \quad R=a+b>1.$$

The inequality is the best possible. *M. Tsuji* (Tokyo)

4252:

Lehner, Joseph. Partial fraction decompositions and expansions of zero. Trans. Amer. Math. Soc. **87** (1958), 130–143.

A function $g(z)$ of the complex variable z is said to be of class (1, 0) if it is an entire function and if the maximum of its absolute value on the circle $|z|=r$ is $<e^{\varepsilon r}$ for every $\varepsilon>0$ if $r>r_0(\varepsilon)$. Let Q be the unit circle, R_1 its interior and R_2 its exterior. The author proves the following. I. Let $g_k(z)$ $(k=1, 2, \cdots)$ be functions of class (1, 0) such that the series $\sum_{k=0}^{\infty}|g_k(n)|$ is convergent for all integral values of n and

$$\limsup_n \left(\sum_{k=0}^{\infty}|g_k(n)|\right)^{1/|n|} \leqq 1. \tag{1}$$

Let further $\{\varepsilon_k\}$ be distinct complex constants of modulus one, and let $\{\alpha_k\}$ be any bounded sequence. Let further

$$\Phi_k(x)=\sum_{n=0}^{\infty} g_k(n)x^n \tag{2}$$

and let Σ be the series

$$\sum_{k=0}^{\infty}\alpha_k\Phi_k(x\varepsilon_k^{-1}).$$

Then the series Σ converges uniformly on any set S at a positive distance from Q and defines two regular functions

$$G_1(x)=\sum_{n=0}^{\infty} a_n x^n, \quad x\in R_1,$$

$$G_2(x)=-\sum_{n=1}^{\infty} a_{-n}x^{-n}, \quad x\in R_2,$$

where for all integral values of n:

$$a_n=\sum_{k=0}^{\infty}\alpha_k\varepsilon_k^{-n}g_k(n). \tag{3}$$

II. Conversely, if Σ converges uniformly on every set S at a positive distance from Q, then functions $\{g_k(z)\}$ exist such that (2), (3), (1) hold.

The author further considers the possibility of continuing the series Σ analytically across the unit circle. If the set $\{\varepsilon_k\}$ is not dense in the unit circle, there is an arc on which the function defined by the series will be regular provided a certain condition is satisfied by the $g_k(n)$. He also considers the inverse problem: given a suitably restricted function $G(x)$ regular in the unit circle, is there a series Σ which represents $G(x)$?—Theorem I includes the case in which a_n is the partition function

$$p(n)=\sum_{h,k}\alpha_{hk}\varepsilon_{hk}^{-n}g_k(n),$$

$$k=1, 2, \cdots; \ h \bmod k, \ (h, k)=1,$$

where $\alpha_{hk}=k^{1/2}\omega_{hk}/\pi 2^{1/2}$, ω_{hk} is a certain $24k$th root of unity, $\varepsilon_{hk}=\exp 2\pi ih/k$ and

$$g_k(z)=\frac{d}{dz}\left\{\frac{\sinh C\lambda(z)/k}{\lambda(z)}\right\}, \quad \lambda(z)=\left(z-\frac{1}{24}\right)^{1/2}.$$

In this case $G_2(z)$ is identically zero.

H. D. Kloosterman (Leiden)

FUNCTIONS OF SEVERAL COMPLEX VARIABLES, COMPLEX MANIFOLDS

See also 4453, 4454.

4253:

Mitchell, Josephine. A complete orthonormal system of homogeneous polynomials on matrix spaces of order 2. Proc. Amer. Math. Soc. **10** (1959), 399–406.

Let Z be a complex 2-by-2 matrix. If we consider the elements of Z as independent complex variables, the condition $I-Z^*Z>0$ defines a domain D in 4-dimensional complex space; this is an irreducible bounded symmetric domain in the sense of E. Cartan. The Bergmann-Šilov boundary B of D consists of the 2-by-2 unitary matrices. For functions analytic in D and having suitable boundary values, $(\varphi, \psi)=\int_B \varphi\bar{\psi}dV$ (dV the Euclidean volume element of B) defines an inner product. It was shown by Bochner [Ann. of Math. (2) **45** (1944), 686–707; MR **6**, 123] for the general case of n-by-n matrices that there exists a complete orthonormal system with respect to this inner product whose elements are homogeneous polynomials. In this paper the author determines such a system explicitly, by direct computation.

A. Korányi (Cambridge, Mass.)

4254:

Aĭzenberg, L. A. Boundary properties of functions analytic in circular regions. Dokl. Akad. Nauk SSSR **125** (1959), 959–962. (Russian)

The author's earlier investigations concerning analyticity of Temliakov integrals [same Dokl. **120** (1958), 935–938; MR **20** #5883] are extended to general Temliakov domains. Some new sufficient conditions for the continuity of the integrals on the boundary are given.

H. Tornehave (Copenhagen)

4255:

Christian, Ulrich. Zur Theorie der Modulfunktionen n-ten Grades. II. Math. Ann. **134** (1958), 298–307.

[For part I, see same Ann. **133** (1957), 281–297; MR **19**, 740.] Let $\mathfrak{Z}_n$ be the space (with complex dimension $k=\frac{1}{2}n(n+1)$) of all complex symmetric $(n\times n)$ matrices Z such that $\Im(Z)$ is positive definite and let Γ_n be Siegel's modular group of degree n. The problem of the compactification of the fundamental domain $\mathfrak{G}_n$ of Γ_n in $\mathfrak{Z}_n$ by adding a suitable complex manifold at infinity was solved by Siegel himself [Comm. Pure Appl. Math. **8** (1955), 677–681; MR **17**, 602] and also by Satake [Proc. Internat. Symposium Algebraic Number Theory, Tokyo and Nikko, 1955, pp. 107–129, Science Council of Japan, Tokyo, 1956; J. Indian Math. Soc. **20** (1956), 259–281; MR **18**, 731, 934]. The latter introduced a topological space $\mathfrak{H}_n=\bigcup_{0\leq\nu\leq n}\mathfrak{F}_\nu$, where $\mathfrak{F}_\nu$ is the quotient space $\mathfrak{Z}_\nu/\Gamma_\nu$ and $\mathfrak{F}_\nu\cap\mathfrak{F}_\mu$ is empty if $\nu\neq\mu$. The present author introduces local coördinates in $\mathfrak{H}_n$ (which was not done yet by Satake) which makes it possible to apply Siegel's method [Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. IIa **1955**, 71–77; MR **17**, 530] for proving that $k+1$ meromorphic functions on $\mathfrak{H}_n$ are algebraically dependent.

H. D. Kloosterman (Leiden)

SPECIAL FUNCTIONS

See also 4325, 4528, 4529, 4562.

4256:

Al-Salam, W. A.; and Carlitz, L. Bernoulli numbers and Bessel polynomials. Duke Math. J. **26** (1959), 437–445.

Proofs are given of identities such as

$$\sum_{k=0}^{n} \frac{(2n+2k+1)!}{(2k)!(2n-2k+1)!} \frac{B_{2m+2k+2}}{(2m+2k+2)!} = \frac{\delta_{mn}(2n+1)!}{(4n+3)!} \quad (0 \leqq m \leqq n)$$

involving the Bernoulli numbers, and of similar identities involving the numbers $C_{n-1}=2^n(1-2^n)B_n/n$. It is also shown that by use of polynomials allied to the Bessel polynomials

$$y_n(z) = \sum_{k=0}^{n} \frac{(n+k)!}{k!(n-k)!} \left(\frac{z}{2}\right)^k$$

the results can be expressed in compact symbolic forms.

A. P. Guinand (Edmonton, Alta.)

4257:

Olkin, Ingram. An inequality satisfied by the gamma function. Skand. Aktuarietidskr. **1958**, 37–39 (1959).

Gurland [same Aktuarietidskr. **1956**, 171–172; MR **20** #1797] has proved that if $\alpha+\delta>0$; $\alpha\neq 0, 1$; $\delta>0$; then

$$(1) \qquad \frac{\Gamma^2(\delta+\alpha)}{\Gamma(\delta)\Gamma(\delta+2\alpha)} \leqq \frac{\delta}{\delta+\alpha^2}.$$

The present author generalizes this result to

$$(2) \qquad \prod_{i=0}^{p-1} \frac{\Gamma^2(\delta+\alpha-\frac{1}{2}i)}{\Gamma(\delta-\frac{1}{2}i)\Gamma(\delta+2\alpha-\frac{1}{2}i)} \leqq \frac{\delta^2(p^2-1)^2+\delta p^4}{\delta^2(p^2-1)^2+\delta p^4+\alpha^2},$$

valid for $\alpha+\delta>(p-1)/2$; $\delta>0$; $p>0$. The proof hinges on finding a lower bound for the variance of a suitable unbiased estimator (Wishart distribution).

It is interesting to note that (1) combined with Gautschi's inequality [J. Math. Phys. **38** (1959/60), 77–81; MR **21** #2067] immediately leads to

$$(3) \qquad \frac{\Gamma(n+\beta)}{\Gamma(n+2\beta)} \leqq \frac{(n+1)^{1-\beta}}{n+\beta^2},$$

where $n=1, 2, \cdots$; $0<\beta<1$.

T. Erber (Chicago, Ill.)

4258:

Tranter, C. J. On the analogies between some series containing Bessel functions and certain special cases of the Weber-Schafheitlin integral. Quart. J. Math. Oxford Ser. (2) **10** (1959), 110–114.

The author investigates the series

$$S_1 = \sum_{s=1}^{\infty} \frac{J_{\nu+2m+k}(\alpha_s)J_\nu(\alpha_s r)}{\alpha_s^k J_{\nu+1}^2(\alpha_s a)},$$

$$S_2 = \sum_{s=1}^{\infty} \frac{J_{\nu+2m+k}(\alpha_s)J_{\nu+2n+k}(\alpha_s)}{\alpha_s^k J_{\nu+1}^2(\alpha_s a)},$$

where ν and k are real, m and n are non-negative integers, $a>1$, $0<r<a$, and the α_s are the positive roots of $J_\nu(\alpha_s a)=0$. If $\nu>-1$ and $k>0$, S_1 can be expressed as a Weber-Schafheitlin integral, and hence explicitly evaluated [Watson, *A treatise on the theory of Bessel functions*, University Press, Cambridge, 1922, sec. 13.4, 13.41]. A corresponding expression for S_2 exists if ν is not a negative integer, k is an integer, $k+m+n>0$, and $\nu+k+m+n>0$. If k fails to be an integer, and otherwise the same conditions prevail, an additional infinite integral will appear in the expression for S_2.

A. Erdélyi (Pasadena, Calif.)

4259:

Kito, Fumiki. On a modified form of Fourier-Bessel integral theorem. Proc. Fac. Engrg. Keio Univ. **10** (1957), 44–47.

The theorem of the title relates to functions of the form $J_n(r\lambda)Y_n(ra)-Y_n(r\lambda)J_n(ra)$ and is practically the same as that found in I. Sneddon's book [*Fourier transforms*, McGraw-Hill, New York, 1951; MR **13**, 29]. The theorem is extended to the case of the functions $J_n(r\lambda)Y_n'(ra)-Y_n(r\lambda)J_n'(ra)$.

N. D. Kazarinoff (Madison, Wis.)

4260:

Pergamenceva, E. D. Integral equations for Lamé functions with period $8K$ in connection with the problem of conformal mapping of a semi-strip with semi-circular cut. Uspehi Mat. Nauk **14** (1959), no. 1 (85), 207–213. (Russian)

Even and odd solutions with period $8K$ of the Lamé equation are shown to satisfy integral equations of the type

$$u(w) = \lambda \int_{-4K}^{4K} g(w, s)u(s)ds,$$

the kernel g being explicitly given in terms of elliptic and hypergeometric functions; λ is a constant for which a series expansion is also given. Similar equations for Lamé functions with period $2K$ and $4K$ are known in the literature.

J. L. Massera (Montevideo)

4261:

Stein, F. Max. Orthogonal functions whose kth derivatives are also orthogonal. SIAM Rev. **1** (1959), 167–170.

It is pointed out that the Hermite, Jacobi, and Laguerre polynomials are not only orthogonal over appropriate intervals and with respect to appropriate weight functions, but the derivatives of arbitrary order, k, have the same property. It is also pointed out that these are the only sets of polynomials which together with their derivatives have this property.

D. C. Lewis, Jr. (Baltimore, Md.)

4262:

Rutishauser, Heinz. Bemerkungen zu einer Arbeit von Al-Salam und Carlitz. Arch. Math. **10** (1959), 292–293.

Al-Salam and Carlitz [Arch. Math. **9** (1958), 412–415; MR **21** #3597] have proved that the two sequences of polynomials $\{U_n(x)\}$, $\{V_n(x)\}$ defined by

$$U_{n+1}(x) = (2n+1)xU_n(x)-U_{n-1}(x) \quad (U_0=1, U_1=x),$$

$$V_{n+1}(x) = (2n+3)xV_n(x)-V_{n-1}(x) \quad (V_0=1, V_1=3x)$$

satisfy orthogonality relations on the real axis; an

equivalent result for $\{U_n(x)\}$ had been obtained earlier by Dickinson [Proc. Amer. Math. Soc. 5 (1954), 946–956; MR 19, 263]. In the present paper these results are obtained rapidly by making use of known theorems on continued fractions. *L. Carlitz* (Durham, N.C.)

4263:

Chatterjea, Santi Kumar. Notes on Jacobi and related polynomials. Bull. Calcutta Math. Soc. **50** (1958), 77–83.

L. Carlitz proved in 1954 that

$$(1-x^2)(P_n'^2-P_{n-1}'P_{n+1}') = n(n+1)(P_n^2-P_{n-1}P_{n+1}),$$

where $P_n=P_n(x)$ is Legendre's polynomial. He also proved that this relation characterizes the system $\{P_n(x)\}$. This result was later extended by various authors to ultraspherical polynomials. The present note gives an extension to general Jacobi polynomials.

G. Szegö (Stanford, Calif.)

4264:

★**Lewin, L. Dilogarithms and associated functions.** Foreword by J. C. P. Miller. Macdonald, London, 1958. xvi+353 pp. 65s.

This book consists of the chapters: I. The dilogarithm; II. The inverse tangent integral; III. The generalized inverse tangent integral; IV. Clausen's integral. V. The dilogarithm of complex argument; VI. The trilogarithm; VII. The higher order functions; VIII. The integration of functions and summation of series; IX. Reference data and tables. The functions treated here are mostly special cases of Lerch's zeta function which, for instance, can be defined for $|z|<1$ by the power series $\Phi(s,v,z)=\sum_0^\infty z^n(v+n)^{-s}$, $v\neq 0,-1,-2,\cdots$. A transformation formula for this function established by Lerch is a generalization of Hurwitz's series for $\sum_{n=0}^\infty (v+n)^{-s}=\zeta(s,v)=\Phi(s,v,1)$. Specialization of the parameters v and s in Φ lead to an increasing number of functional equations. For instance Gauss's dilogarithm is defined as

$$\mathrm{Li}_2(z) = \Phi(1,1,z) = -\int_0^z z^{-1}\log(1+z)\,dz$$

and a large number of other special cases (among them the Fermi-Dirac functions) have become increasingly important within the last 10 years or so. Applications cover such diversified fields as electrical network theory, microwave theory (especially wave guide and cavity resonator phenomena), quantum statistics, molecular physics, etc. Consequently, many properties and functional equations of such special cases of Φ have been "rediscovered" many times. It is therefore a pleasure to report that the book under review should prevent any confusion and duplicity of work as far as results up to now are concerned. Unfortunately (or rather fortunately) the material presented here is so vast that it is impossible to give a detailed account. Regarding the applied scientist there is no need to elaborate on the desirability and necessity of such a book. But this reviewer had the feeling that the care and enthusiasm displayed here went beyond the underlying "practical" purpose. This book should prevent the falling into oblivion of a strangely fascinating island of 19th century mathematics full of bizarre and striking results. *F. Oberhettinger* (Corvallis, Ore.)

ORDINARY DIFFERENTIAL EQUATIONS

See also 4211, 4333, 4640, 4737, 4763.

4265:

Conti, Roberto. Sistemi differenziali ordinari con condizioni lineari. Ann. Mat. Pura Appl. (4) **46** (1958), 109–130.

In this paper conditions are given which guarantee the existence of at least one solution of the nonlinear differential equation $dx/dt=A(t,x)x+a(t,x)$ satisfying the linear boundary condition $\int_\gamma^\delta dF(t)x(t)=c$, where $x=x(t)$ and $a(t,x)$ are real n-component vectors; c is a given real constant n-component vector; and $A(t,x)$ and the $F(t)$ of bounded variation are given n by n square matrices. The results obtained provide, as indicated, an extension of the work of W. M. Whyburn and others. Uniqueness of the solution is also established in certain special cases. *H. L. Turrittin* (Minneapolis, Minn.)

4266:

Haimovici, Adolf. Une démonstration globale du théorème d'existence pour les intégrales de certaines équations différentielles. An. Şti. Univ. "Al. I. Cuza" Iaşi. Secţ. I. (N.S.) **4** (1958), 53–60. (Romanian and Russian summaries)

The theorem that each solution of a differential equation $x'=f(t,x)$ can be prolonged to the boundary of the domain of continuity of $f(t,x)$ is proved by a method shown to be valid in a Banach space of countable base. A crucial tool is the theorem that a bounded continuous mapping of a closed subset of one Banach space into another Banach space can be extended to the whole space, while preserving the bound. *W. Kaplan* (Ann Arbor, Mich.)

4267:

Consiglio, A. Trasformazioni semicanoniche e trasformazioni canoniche. Atti Accad. Gioenia Catania (6) **11** (1957/58), 173–183. (French, English and German summaries)

By a semi-canonical transformation the author means a transformation by which a given system S of differential equations may be written in the Hamiltonian form. The set G of such transformations, arising from a particular S, does not (unlike the properly canonical transformations, which have no reference to a particular S) in general form a group. Moreover the set G is vacuous unless S is derivable from a certain type of variational principle. The author develops some rather complicated but straightforward analytical conditions to be satisfied by a transformation if it is to be semi-canonical.

D. C. Lewis, Jr. (Baltimore, Md.)

4268:

Makai, E. A class of systems of differential equations and its treatment with matrix methods. II. Publ. Math. Debrecen **5** (1958), 269–274.

This is a continuation of the author's paper in the same Publ. **5** (1957), 5–37 [MR **19**, 1051]. The author considers differential equations of the type

$$(1)\quad P_n(x)y^{(m)}+P_{n-1}(x)y^{(m-1)}+\cdots+P_{n-m}(x)y=0,$$

where P_{n-i} is a polynomial of degree $\leq n-i$ $(i=0,1,\cdots,m)$, and P_n has distinct roots $a_1,\cdots,a_n$. He presents

an algorithm which assigns, to each equation (1), a system of type

(2) $(x-a_i)y_i' = \sum\{a_{ij}y_j : j=1, \cdots, n\}$ $(i = 1, \cdots, n)$,

(with the a_{ij} constant and satisfying certain additional conditions), such that y is a solution of (1) if and only if there exists a solution $(y_1, \cdots, y_n)$ of (2) with $y_1 = y$. This result is applied to show that each solution of an equation of type

(3) $$y^{(m)} + f_{m-1}(x)y^{(m-1)} + \cdots + f_0(x) = 0,$$

where the f_j are regular on a disk C, may be approximated uniformly on C by functions y_1 such that $(y_1, \cdots, y_n)$ is a solution of a system (2) with large n.

W. Strodt (New York, N.Y.)

4269:

Barrett, L. C.; and Wylie, C. R., Jr. Orthogonality conditions in systems with both continuous and lumped parameters. Univ. Nac. Tucumán. Rev. Ser. A **12**, 73–80 (1959).

In this paper the authors discuss eigenvalue problems arising from applications in which the boundary conditions involve the eigenvalue parameter and the equation may be of the form $Lx = \lambda Mx$. They include cases where conditions are imposed at interior points of the interval and indicate how these give rise to eigenfunctions which are orthogonal with respect to an inner product of the form

$$(u, v) = \sum_{j=0}^{n} \int_a^b \frac{d^j u}{dx^j}\frac{d^j v}{dx^j}\, dP_j(x),$$

where the integrator functions $P_j(x)$ may be discontinuous.

R. R. D. Kemp (Kingston, Ont.)

4270:

Langer, Rudolph E. The asymptotic solutions of a linear differential equation of the second order with two turning points. Trans. Amer. Math. Soc. **90** (1959), 113–142.

This paper provides the leading terms of the asymptotic representations of solutions of

$$\frac{d^2y}{ds^2} + \{\lambda^2\theta^2(s) + X(s, \lambda)\}y = 0, \quad a \leqq s \leqq b,$$

for large complex values of the parameter λ^2, in the special case for which $\theta^2(s)$ is real, and has precisely two simple zeros in the interval (a, b). $X(s, \lambda)$ is bounded in s, λ and integrable in s. The zeros of $\theta^2(s)$ are called turning points since $\theta^2(s)$ changes sign as s increases through such a zero, and the behavior of the solutions changes from exponential to oscillatory. After a suitable normalization, the equation is compared with Weber's equation

$$\frac{d^2\omega}{dz^2} + (2k + 1 - z^2)\omega = 0,$$

which takes the form $d^2u/ds^2 + \lambda^2\theta^2(s)u = 0$ after a similar normalization. The leading terms of the asymptotic representations are given explicitly in terms of special Weber functions for various regions of the complex λ plane. If $\lambda^2\theta^2(s)$ is positive between the turning points there exists a denumerable infinity of special values (eigenvalues) λ_n, for which the solutions remain uniformly bounded as $\lambda_n \to \infty$, while for other values of λ, bounded away from λ_n, the solutions are unbounded.

G. E. Latta (Stanford, Calif.)

4271:

Wintner, Aurel. The Schwarzian derivative and the approximation method of Brillouin. Quart. Appl. Math. **16** (1958), 82–86.

Let $f(t)$ be twice continuously differentiable and nonzero for large t. Define

$$Y(t) = f(t)^{-1/2} \exp\left\{i \int_{t_0}^{t} f(s)ds\right\};$$

then $Y(t)$ satisfies the identity

$$Y'' + [1 - F(t)]f(t)^2 Y = 0$$

with $4F(t) = 3f'^2/f^4 - 2f''/f^3$. Brillouin used this identity in order to study the asymptotics of solutions of (1) $y'' + f(t)^2 y = 0$ [same Quart. **6** (1948), 166–178; **7** (1950), 363–380; MR **10**, 252; **11**, 721]. In a previous paper [ibid. **15** (1957), 428–430; MR **19**, 1051] the author proved the general criterion: Let $g(t)$ and $h(t)$ be continuous for large t and consider the differential equations $y'' + g(t)y = 0$, $x'' + h(t)x = 0$. If $y_1(t)$, $y_2(t)$ are two linearly independent solutions of the first equation, and if $h(t)$ is so "close" to $g(t)$ that $\int^\infty (|y_1|^2 + |y_2|^2)|g - h|dt < \infty$, then the two differential equations are asymptotically equivalent; that is, there exists a solution system $x_1(t)$, $x_2(t)$ of the second equation such that $x_1 \sim y_1$, $x_2 \sim y_2$, $x_1' \sim y_1'$, $x_2' \sim y_2'$ as $t \to \infty$. In the present paper, this criterion is combined with Brillouin's identity in order to prove: If $f(t)$ is positive and twice continuously differentiable for large t, if $\int^\infty |F(t)|f(t)dt < \infty$, then the general solution of (1) is asymptotically equivalent to the real and imaginary parts of $Y(t)$. This is an improvement of a previous result of the author [Phys. Rev. (2) **72** (1947), 516–517; MR **9**, 144].

M. Schiffer (Stanford, Calif.)

4272a:

Iglisch, Rudolf. Der Resonanzbegriff bei linearen gewöhnlichen Differentialgleichungen zweiter Ordnung. Arch. Rational Mech. Anal. **3**, 179–186 (1959).

4272b:

Iglisch, Rudolf. Der Resonanzfall bei nichtlinearen gewöhnlichen Differentialgleichungen zweiter Ordnung. Arch. Rational Mech. Anal. **3**, 187–193 (1959).

In the first paper the author shows that the equation

$$L[y] \equiv y'' + a(t)y' + b(t)y = f(t)$$

with a, b, and f continuous and periodic of period P has bounded solutions if the adjoint to $L[y] = 0$ has either periodic solutions z with period P for which $\int_0^P zfdt = 0$ or no periodic solutions with period P at all. Otherwise, every solution y has arbitrarily large absolute values for $|t|$ large enough. The former result is contained in a theorem of Fite [Ann. of Math. (2) **28** (1926), 59–64] for nth order linear equations; the latter is a special case of similar results for first order linear systems, which can be proved more directly.

The second paper concerns the equation $z'' + g(z, z', t) = f(t) + \beta F(t)$ with g of class C^2 and g in t, f, and F periodic of period P. If y is a periodic solution with period P for $\beta = 0$ and if the variation equation $L[\varphi] = 0$ has no periodic solution with period P, the author proves that there are solutions z for which $|z - y| \leq |\beta|C$ for all t. If the adjoint to $L[\varphi] = 0$ has a periodic solution u with period

P for which $\int_0^P uF\,dt \neq 0$ then every solution z is such that $|z-y|$ takes on values of order $\sqrt{|\beta|}$ with increasing t. *H. A. Antosiewicz* (Los Angeles, Calif.)

4273:

Yoshizawa, Taro. Note on the equi-ultimate boundedness of solutions of $x'=F(t,x)$. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **31** (1958), 211–217.

The author continues his analysis of uniform boundedness, uniform ultimate boundedness, and equi-ultimate boundedness for the solutions of (1) $dx/dt=F(t,x)$ [cf. same Mem. **29** (1955), 275–291; **30** (1957), 217–226; MR **20** #4679, #4680]. Here x is a real n-vector and $F(t,x)$ is a vector field continuous in Δ: $0\leq t<\infty$, $\|x\|<\infty$, where $\|x\|=[\sum_{i=1}^n x_i^2]^{1/2}$. Also assume the uniqueness of solutions through an initial point (t_0,x_0). A typical result is the following. If the solutions of (1) are uniformly bounded and ultimately bounded, they are equi-ultimately bounded. The author gives necessary and sufficient conditions for this occurrence in terms of the existence of certain positive functions which relate to the growth of $F(t,x)$. *L. Markus* (Minneapolis, Minn.)

4274:

Foddis, Irene. Sull'esistenza di soluzioni periodiche per una particolare classe di equazioni differenziali ordinarie del secondo ordine. Ann. Univ. Ferrara. Sez. VII. (N.S.) **7** (1957/1958), 73–82.

The author considers the equation

$$\ddot{x}+P(t,x,\dot{x})x=Q(t,x,\dot{x},\ddot{x}),$$

where P and Q satisfy certain simple continuity conditions and are periodic with respect to t with the period ω, and gives a detailed proof of the following result which is ascribed to Volpato. If there exists a constant $k<1$ and continuous periodic functions $p(t)$ and $P(t)$ such that $\omega\int_0^\omega P(t)\,dt\leq 4$, $0\leq p(t)\leq P(t,x,\dot{x})\leq P(t)$, $|Q(t,x,\dot{x},\ddot{x}_1)-Q(t,x,\dot{x},\ddot{x}_2)|\leq k|\ddot{x}_1-\ddot{x}_2|$, for all real x and $\dot{x}$, the differential equation has a solution $x(t)$ which is periodic with the period ω. *L. A. MacColl* (New York, N.Y.)

4275:

Opial, Z. Sur l'existence des solutions périodiques de l'équation différentielle du second ordre. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 71–75.

Let $f(t,u,v)$ be continuous, periodic in t with period 1 and non-decreasing in u. If Cauchy's problem for $u''=f(t,u,u')$ always has a unique solution and there is at least one solution such that u, u' are bounded for $t\geq 0$, a periodic solution of period 1 exists [cf. a theorem of the reviewer in Duke Math. J. **17** (1950), 457–475; MR **12**, 705]. Any solution bounded in $(-\infty,\infty)$ is periodic and any two periodic solutions coincide modulo a constant; for any solution $v(t)$ bounded in $(0,\infty)$ there is a periodic solution $u(t)$ with $v(t)-u(t)\to 0$, and similarly for $(-\infty,0)$; if f is strictly increasing in u, the periodic solution is unique. *J. L. Massera* (Montevideo)

4276:

Barbălat, I.; et Halanay, A. Solutions périodiques des systèmes d'équations différentielles non linéaires. Rev. Math. Pures Appl. **3** (1958), 395–411.

The authors obtain the existence of periodic solutions of certain forced nonlinear systems which are in a sense close to linear systems. The hypotheses are usually too involved to be given here completely. They first consider (1) $\dot{x}=A(t)x+f(x,t)$, where x is an n-vector, $A(t)$ is a continuous matrix, and $f(x,t)$ is a continuous vector, both defined for all (x,t), and both periodic in t of period T. It is assumed that (2) $\dot{x}=A(t)x$ has no nontrivial periodic solution of period T, and that $|f(x,t)|\leq b|x|+M$, where b is a sufficiently small positive number depending on $A(t)$. Then (1) is shown to have at least one solution of period T. The proof is by Banach space methods. If (2) has nontrivial periodic solutions, it is necessary to impose other, highly restrictive, conditions on $f(x,t)$. Other cases, in which A depends on x as well as on t are considered.

The case of large forcing is considered with (3) $\dot{x}=A(t)x+\lambda e(t)+f(x,t)$, where $e(t)$ is an n-vector of period T, and A and f are as in (1). The coefficient λ is considered large and positive. It is shown that under appropriate conditions (3) has a unique solution of period T if λ is sufficiently large. If the trivial solution of (2) is uniformly asymptotically stable, the same is true of the periodic solution of (3) for sufficiently large λ, and the domain of attraction increases with λ. The existence in this case is a generalization of a result of the reviewer for second-order scalar systems with A and f independent of x [Duke Math. J. **24** (1957), 235–247; MR **19**, 36]. The paper concludes with some applications of the results to systems which represent certain servomechanisms with limiting. *W. S. Loud* (Madison, Wis.)

4277:

Barbălat, I. Applications du principe topologique de T. Ważewski aux équations différentielles du second ordre. Ann. Polon. Math. **5** (1958/59), 303–317.

A number of results on the existence of bounded, periodic and almost-periodic solutions of $\ddot{x}=\phi(x,\dot{x},t)$ are proved; the following are typical. 1. If ϕ is defined for $a<x<b$, $-\infty<\dot{x},t<\infty$, and if $\phi(x_0,0,t)<0$, $\phi(x_1,0,t)>0$ for some x_0,x_1, $a<x_0<x_1<b$, and any t, there exist an infinity of solutions bounded in the future and an infinity bounded in the past. 2. If certain boundedness assumptions on ϕ and ϕ_x are added, not only x but also $\dot{x}$ is bounded and there is at least one solution for which x and $\dot{x}$ are bounded both in the past and in the future. 3. If moreover ϕ is almost-periodic in t, uniformly with respect to x, $\dot{x}$, and $\phi_x\geq m>0$, the bounded solutions are almost-periodic. 4. If $p(t)$ is almost-periodic, $g(x)$, $g'(x)$ are continuous, $g'(x)<0$, $g(x)\to\mp\infty$ for $x\to\pm\infty$, and if $f'(y)$ satisfies certain boundedness conditions, $\ddot{x}+f(\dot{x})+g(x)=p(t)$ has almost-periodic solutions whose derivatives are also almost-periodic. 5. If $p(t)$ is periodic of period T, $g(x)\to\mp\infty$ for $x\to\pm\infty$, $|g(x)|\leq\omega(|x|)$, $|\int_0^x f(\xi)d\xi|\leq\omega(|x|)$, with $\int_a^\infty dz/\omega(z)=\infty$, then $\ddot{x}+f(x)\dot{x}+g(x)=p(t)$ has solutions of period T. *J. L. Massera* (Montevideo)

4278:

★**Seifert, George. Rotated vector fields and an equation for relaxation oscillations.** Contributions to the theory of nonlinear oscillations, Vol. IV, pp. 125–139. Annals of Mathematics Studies, no. 41. Princeton University Press, Princeton, N.J., 1958. ix+211 pp. $3.75.

Let $\dot{x}=P(x,y,\alpha)$, $\dot{y}=Q(x,y,\alpha)$, the singular points being isolated and independent of α, P, Q continuous in R^3,

locally Lipschitzian in (x, y); it is assumed that P_x, Q_x exist and $0 \leq \Delta\theta = \theta(x, y, \alpha_2) - \theta(x, y, \alpha_1) < \pi$ for $\alpha_1 < \alpha_2$, where $\theta = \arg(P, Q)$; if $\Delta\theta = 0$ for some $\alpha_2 > \alpha_1$, (x, y) is said to be an α_1-stationary, otherwise an α_1-rotation point, and it is assumed that on any finite arc of any α-trajectory there is at most a finite number of α-stationary points. Theorems: (1) The region bounded by finite arcs of an α_0- and an α_1-trajectory which have common endpoints contains a critical point. (2) Any α_0-cycle is disjoint with any α_1-cycle. (3) If N is an outer neighborhood of an α_0-cycle which is (asymptotically) stable from outside, there exists an α_1 such that, for each α, $\alpha_0 < \alpha \leq \alpha_1$, N contains an α-cycle stable from outside and an α-cycle stable from inside (both may coincide); other general conditions for the existence of stable and orbitally stable cycles are given.

As an application, new conditions for the existence of limit cycles of $\ddot{x} + f(x, \dot{x})\dot{x} + g(x) = 0$ are given, which cover cases where Levinson-Smith criteria do not apply as, for instance, $f(x, y) = x^4(|y| + 1)^{-1} - 1$, $g = \frac{1}{2}x$. Cf. G. F. D. Duff [Ann. of Math. (2) **57** (1953), 15–31; MR **14**, 751].

J. L. Massera (Montevideo)

4279:

Molčanov, N. N. A proof of the existence of an integrating factor in the neighborhood of a singular point. Mat. Sb. (N.S.) **47 (89)** (1959), 265–270. (Russian)

Let $X_i = X_i(x_1, \cdots, x_n)$, $i = 1, \cdots, n$, be analytic functions in a hypersphere G in E_n of sufficiently small radius with center at some point $P_0(x_1^0, \cdots, x_n^0)$. Let P_0 be a singular point of the autonomous system (1) $dx_i/ds = X_i(x_1, \cdots, x_n)$, $i = 1, \cdots, n$. The following result, of interest by itself, is first proved. Consider the linear partial differential equation of first order (2) $\sum_{i=1}^n X_i \, \partial f/\partial x_i = 0$, where X_i are as in (1); note that in general (2) will not have analytic solutions in a neighborhood of P_0. Theorem: There exist two functions $u_1(x_1, \cdots, x_n, z)$ and $u_2(x_1, \cdots, x_n, z)$ satisfying the conditions (i) u_1 and u_2 are analytic functions of $x_1, \cdots, x_n$ provided $|z|$ is sufficiently small and the point $(x_1, \cdots, x_n) \in G$ and (ii) the relation $u_1(x_1, \cdots, x_n, z) = c$ will be a solution of (2) if z, regarded as an implicit function of $(x_1, \cdots, x_n)$ is determined from the equation $u_2(x_1, \cdots, x_n, z) = 0$. The proof uses the Cauchy-Kowalevski theorem applied to equation (3) $\sum_{i=1}^n X_i \partial u/\partial x_i + Z \partial u/\partial z = 0$, where X_i are as in (1), (2) and $Z = Z(x_1, \cdots, x_n, z)$ is an arbitrary analytic function of its arguments which has no zeros in G for all $|z|$ sufficiently small. The existence of $n-1$ functionally independent solutions of (2) [equivalently $n-1$ independent integrals of (1)] follows from the existence of n solutions $u_i = \phi_i(x_1, \cdots, x_n, z)$, $i = 1, \cdots, n$, of (3) for which the Jacobian $\partial(\phi_1, \cdots, \phi_n)/\partial(x_1, \cdots, x_n) \not\equiv 0$ in G for $|z|$ sufficiently small and the above theorem.

The existence of an integrating factor of Jacobi for the system (1) in a neighborhood of P_0 is then deduced by applying the theory of continuous groups of transformations. The non-vanishing of the above Jacobian and the fact that the Weierstrass preparation theorem may be used to write the equation $u_2(x_1, \cdots, x_n, z) = 0$ in the equivalent form $a_0 z^j + a_1 z^{j-1} + \cdots + a_n = 0$, where j is a positive integer and $a_0, \cdots, a_n$ are analytic functions of $x_1, \cdots, x_n$ in G, are needed here. Note that the function z determined from the above equation and hence also the corresponding solutions of (2) will, in general, have singularities in G. The reviewer observed a slight error in the example on page 267; the quantity $x^2 + y^2$ in the denominator should appear as $(x^2 + y^2)^2$.

J. A. Nohel (Atlanta, Ga.)

4280:

Hronec, J. Die Bewegungen mit n Freiheitsgraden, wo die kinetische und die potentielle Energie mit der quadratischen Form gegeben ist. Acta Fac. Nat. Univ. Comenian. **3**, 1–15 (1958). (Slovak and Russian summaries)

An exposition of well-known facts about linear systems of ordinary differential equations with constant coefficients with particular attention to stable linear conservative dynamical systems. *D. C. Lewis, Jr.* (Baltimore, Md.)

4281:

Peixoto, M. M. Some examples on n-dimensional structural stability. Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 633–636.

A real C^1-differential system

$$(1) \qquad \dot{x}_i = \frac{dx_i}{dt} = X_i(x_1, \cdots, x_n) \quad (i = 1, 2, \cdots, n)$$

is structurally stable in the unit ball B^n in case (i) the vector field X everywhere points inward across the boundary S^{n-1} of B^n, and (ii) there exists $\delta > 0$ such that whenever

$$(2) \qquad \dot{x}_i = Y_i(x_1, \cdots, x_n) \in C^1 \text{ in } B^n$$

satisfies

$$\max_{i,j;x \in B^n} \left[|X_i - Y_i| + \left| \frac{\partial X_i}{\partial x_j} - \frac{\partial Y_i}{\partial x_j} \right| \right] < \delta,$$

there exists a homeomorphism T of B^n onto itself mapping the trajectories of (1) onto those of (2). Often T is required to be an ε-homeomorphism and then $\delta = \delta(\varepsilon)$. The author indicates the construction of infinitely many different types of structurally stable systems in B^n, for $n \geq 2$. These examples have limit sets which are saddle- and node-type isolated critical points and stable isolated periodic solutions. *L. Markus* (Minneapolis, Minn.)

4282:

Kopeć, Józef. On linear differential equations in Banach spaces. Zeszyty Nauk. Uniw. Mickiewicza. Mat.-Chem. **1** (1957), 3–7. (Polish. Russian and English summaries)

The author deals with almost periodic Banach-space valued solutions of the differential equation

$$L(x) \equiv x^{(n)} + a_1 x^{(n-1)} + \cdots + a_n x = y(t),$$

with constant coefficients. The function $x(t)$ is called (strongly) almost periodic if for every $\varepsilon > 0$ it admits a relatively dense set of ε-almost periods; the function $x(t)$ is called weakly almost periodic if for every linear functional ξ the function $\xi(x(t))$ is almost periodic. It is shown: (1) A weakly almost periodic function is almost periodic if and only if it is compactly valued. (2) The primitive function of an almost periodic function is almost periodic if and only if it is compactly valued. (3) Let $y(t)$ be almost periodic; any compactly valued solution of the equation $L(x) = y(t)$ is almost periodic. It is shown that all these results are no longer true when the hypothesis that the

involved functions are compactly valued is replaced by the hypothesis of boundedness.

A. Alexiewicz (Zbl 80, 330)

4283:

Mlak, W. Limitations and dependence on parameter of solutions of non-stationary almost linear differential operators equations. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 677–682.

Il lavoro riguarda lo studio di equazioni differenziali del tipo

$$\frac{dx}{dt} = A(t)x + f(t, x),$$

dove $A(t)$ è un operatore lineare chiuso con dominio in uno spazio di Banach E. L'A. dà limitazioni "a priori", teoremi di dipendenza continua dai valori iniziali e da parametri, teoremi di esistenza "in grande" e di derivazione rispetto a parametri. Le dimostrazioni dei numerosi enunciati sono solo brevemente accennate. L'A. si ricollega al fondamentale lavoro di Kato [J. Math. Soc. Japan **5**, (1953), 208–234; MR **15**, 437]. Egli utilizza in particolare il lemma 1 di questo lavoro per poter applicare alla funzione $|x(t)|$ (norma di $x(t)$) i teoremi di confronto validi per le equazioni differenziali ordinarie. Riportiamo un enunciato tipico: Ipotesi: (1) Per ogni $t \in (0, \alpha]$ $(\alpha > 0)$ l'operatore $A(t)$, definito in una varietà lineare di E densa in E, abbia un risolvente $R(\lambda, A(t))$ per λ abbastanza grande e sia inoltre $|\lambda R(\lambda, A(t))| \leq 1$. (2) $x(t)$ sia continua e soddisfa l'equazione $D_+x(t) = A(t)x(t) + f(t, x(t))$ in $(0, \alpha]$ con eccezione, al più, per un sottoinsieme numerabile di valori. (3) Sia $|f(t, x(t))| \leq \sigma(t, |x|)$, essendo σ una funzione non negativa, continua e tale che, per ogni $\eta \geq 0$, l'integrale massima $\omega(t, \eta)$ dell'equazione $\omega' = \sigma(t, \omega)$ individuato dalla condizione iniziale $\omega(0, \eta) = \eta$ esista in tutto l'intervallo $[0, \alpha]$. Tesi: vale la diseguaglianza $|x(t)| \leq \omega(t, |x(0)|)$ in tutto l'intervallo $[0, \alpha]$. *G. Prodi* (Trieste)

4284:

Corduneanu, C. Équations différentielles dans les espaces de Banach. L'applicabilité du principe topologique de T. Ważewski. Acad. R. P. Romîne. Fil. Iaşi. Stud. Cerc. Şti. Mat. **9** (1958), no. 1, 101–111. (Romanian. Russian and French summaries)

L'A. étend aux espaces de Banach une méthode de T. Ważewski concernant le comportement asymptotique des solutions des équations différentielles [Ann. Soc. Polon. Math. **20** (1947), 279–313; MR **10**, 122]. Soit (1) $dx/dt = f(t, x)$, $x(t_0) = x_0$, l'équation considérée. On suppose la fonction f localement sommable au sens de Bochner par rapport à t avec x fixe et

$$\|f(t, x_1) - f(t, x_2)\| \leq \psi_D(t)\|x_1 - x_2\|$$

où la fonction ψ dépend de l'ensemble borné D, contenu dans le domaine Ω de définition de f. On démontre l'existence et l'unicité de la solution de l'équation intégrale correspondante et la possibilité de prolongement jusqu'à la frontière du domaine. Soit $\omega \subset \Omega$ un ensemble ouvert tel que l'ensemble des points par lesquels les solutions de l'équation (1) sortent de ω, la sortie étant "forte" au sens de A. Bielecki [Bull. Acad. Polon. Sci. Cl. III **4** (1956), 493–495; MR **18**, 897] possède une certaine retracte. Alors ω contient une solution définie pour $t \geq t_0$. *G. Marinescu* (Bucharest)

PARTIAL DIFFERENTIAL EQUATIONS

See also 4307, 4308, 4371, 4549, 4637.

4285:

Sobolev, S. L. Certain Soviet articles on the application of functional analysis to differential equations. Advancement in Math. **3** (1957), 577–593. (Chinese)

This is a translation of a lecture given by Sobolev on Sept. 6th 1955 in Prague. The main topics touched upon in the lecture are as follows. (1) Functional space W_p^l, basic theorems and their important corollaries (Sobolev, Il'in and Nikol'skiĭ; (2) applications of the spaces W_p^l to variational problems, compactness of the minimizing sequences and boundary values of the limit functions; (3) strong elliptic equations by operator methods (Ladyženskaya, Višik, Guseva), elliptic equations with degeneration on the boundary (Višik, Mihlin); (4) continuity and differentiability properties of solutions (in W_p^l) of Cauchy problems of hyperbolic equations, for example, solutions in W_2^s $(s \geq 1)$ of the wave equation with initial values u_0 in W_2^s and u_1 in W_2^{s-1}; (5) mixed problems of hyperbolic and parabolic equations (Višik, Ladyženskaya).

Yu Why Chen (New York, N.Y.)

4286:

Il'in, V. A. On uniform convergence of expansions in eigenfunctions on a whole closed region. Mat. Sb. N.S. **45** (87) (1958), 195–232. (Russian)

This paper is concerned with the uniform convergence of eigenfunction expansions arising from $\Delta u + \lambda u = 0$ on a region g in N-space together with homogeneous boundary conditions (1) $u = 0$, (2) $du/dn = 0$, or (3) $du/dn + h(S)u = 0$ on the boundary Γ of g. This is done by proving essentially that a suitable iteration of the Green's function has a uniformly convergent eigenfunction expansion and then using this. If $\lambda_1, \lambda_2, \cdots$ are the non-zero eigenvalues and $u_1(P), u_2(P), \cdots$ the corresponding eigenfunctions the author proves (under suitable restrictions on Γ) that $\sum_{i=1}^{\infty} u_i^2(P)\, \lambda_i^{-N/2-\varepsilon/2}$ converges uniformly on $g \cup \Gamma$ for $\varepsilon > 0$. This allows him to show that if $\Delta^k f$ exists and satisfies the boundary conditions in a generalized sense for suitable k, then the eigenfunction expansion of f converges uniformly on $g \cup \Gamma$. He also obtains with this technique an estimate of the order of the remainder in the expansion after a finite number of terms.

R. R. D. Kemp (Kingston, Ont.)

4287:

Bergendal, Gunnar. Convergence and summability of eigenfunction expansions connected with elliptic differential operators. Medd. Lunds Univ. Mat. Sem. **15** (1959), 63 pp.

The spectral function $e(\lambda, x, y)$ related to the problem of an elliptic differential operator in real n-space is the generalization of the function $e(\lambda, x, y) = \sum_{\lambda_k < \lambda} \bar{\phi}_k(x)\phi_k(y)$ in the case where the spectrum is discrete and the eigenfunctions are ϕ_k. The partial sum function for a function $f(x)$ of L^2 is $e(\lambda, f, y) = \int e(\lambda, x, y) f(x) dx$. Improved estimates are obtained for the asymptotic behaviour of various partial derivatives of these functions as $\lambda \to \infty$, and applications are made to problems of localization, convergence and summability. The analysis is based on the methods of Ganelius, Gårding and Pleijel.

E. C. Titchmarsh (Oxford)

4288:

Shimoda, Setsuro. Comparison theorems and various related principles in the theory of second order partial differential equations of elliptic type. Sûgaku **9** (1957/58), 153–166. (Japanese)

This is an expository article on the maximum principle of second order differential operators. The outline follows that of Tautz [Math. Ann. **118** (1943), 733–770; MR **6**, 3]. Tautz's eight axioms are quoted. The author thoroughly discusses systems of equivalent axioms, relations among them and their direct consequences. Next, fairly extensive study is made of E. Hopf's maximum principle. The author gives an extension of Hopf's result [Proc. Amer. Math. Soc. **3** (1952), 791–793; MR **14**, 280] with non-vanishing term. Finally, some comments are made about non-linear cases, on cases with a parameter, and on the case when the operator involves a generalized Laplacian.

H. Yamabe (Minneapolis, Minn.)

4289:

Kapilevič, M. B. The theory of degenerate elliptic differential equations of Bessel class. Dokl. Akad. Nauk SSSR **125** (1959), 719–722. (Russian)

The author considers solutions of the equation

$$\Delta u+(k/x)u_x+(m/y)u_y+(n/z)u_z+F(r)u=0$$

$$(r=(x^2+y^2+z^2)^{1/2};\ 0<k,m,n<1)$$

which are defined on $D_1=[x\geq 0]$, on $D_2=D_1\cap[y\geq 0]$ or on $D_3=D_2\cap[z\geq 0]$. Under the assumption that either u or $\partial u/\partial n$ vanishes on ∂D_i he investigates certain mean values of u extended over $[x^2+y^2+z^2=r^2]\cap D_i$ $(i=1,2,3)$.

A. Huber (Muenchenstein)

4290:

Gagliardo, Emilio. Un'osservazione sul problema di Dirichlet per un'equazione lineare ellittica del secondo ordine. Rend. Accad. Sci. Fis. Mat. Napoli (4) **25** (1958), 66–80.

Let Ω be a bounded domain with boundary Γ of class $A^{(2,\lambda)}$ [in C. Miranda's notation, *Equazioni alle derivate parziali di tipo ellittico*, Springer, Berlin-Göttingen-Heidelberg, 1955; MR **19**, 421; p. 2]. Denote by $W_p^r(\Omega)$ the space of functions, whose partial derivatives of order $\leq r$, taken in the sense of distributions, belong to $L_p(\Omega)$ $(p>1)$, and define similarly the space $W_p^r(\Gamma)$. Consider the boundary value problem $L(u)=f$ in Ω, $\gamma(u)=\varphi$ on Γ, where the right-hand side f and the boundary values φ belong to $L_p(\Omega)$ and $W_p^2(\Gamma)$, respectively. $L(u)$ denotes an elliptic differential operator:

$$L(u)\equiv\sum_{i,j=1}^{n}a_{ij}(x)\frac{\partial^2 u}{\partial x_i\partial x_j}+\sum_{i=1}^{n}b_i(x)\frac{\partial u}{\partial x_i}+c(x)u$$

with $\sum a_{ij}(x)\xi_i\xi_j\geq c_0\sum\xi_i^2$, $c_0>0$. The first derivatives of the a_{ij} and the b_i are assumed to be Lipschitz-continuous in $\bar{\Omega}$, and c is a nonpositive bounded measurable function. The author proves existence and uniqueness of a solution u in $W_p^2(\Omega)$ satisfying the a priori estimation

$$\|u\|_{W_p^2(\Omega)}\leq c_1\{\|f\|_{L_p(\Omega)}+\|\varphi\|_{W_p^2(\Gamma)}\}.$$

The constant c_1 depends on Ω, p, c_0, $\operatorname{Max}|a_{ij}(x)|$, $\operatorname{Max}|b_i(x)|$, $\sup|c(x)|$, and the Lipschitz-coefficients of $\partial a_{ij}/\partial x_k$ and b_i. This result is a generalization of theorems of G. Stampacchia [Ann. Mat. Pura Appl. (4) **33** (1952), 211–238; MR **14**, 483; $p=2$], O. A. Ladyženskaya [Dokl. Akad. Nauk SSSR **79** (1951), 723–725; MR **14**, 280; $p=2$ and homogeneous boundary conditions], and A. I. Koševlev [Dokl. Akad. Nauk SSSR **105** (1955), 22–25; MR **17**, 620; $p>n$; see also ibid. **116** (1957), 542–544; MR **20** #3378; and, for the other theorem, Mat. Sb. (N.S.) **38** (80) (1956), 359–372; MR **17**, 1213]. A similar theorem is proved for self-adjoint differential equations with

$$L(u)\equiv\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right)+c(x)u.$$

Here the $a_{ij}(x)$ are assumed to be Lipschitz-continuous in $\bar{\Omega}$ and the constant c_1 in the a priori estimate depends on Ω, p, c_0, $\operatorname{Max}|a_{ij}(x)|$, $\sup|c(x)|$, and the Lipschitz-coefficients of the a_{ij}.

J. C. C. Nitsche (Minneapolis, Minn.)

4291:

Lehman, R. Sherman. Developments at an analytic corner of solutions of elliptic partial differential equations. J. Math. Mech. **8** (1959), 727–760.

The differential equations under consideration are of the type

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+A\frac{\partial u}{\partial x}+B\frac{\partial u}{\partial y}+Cu=f,$$

where A, B, C and f are analytic functions of the two variables x, y in a neighborhood of the origin. Let D be an open domain whose boundary contains two analytic arcs Γ_1 and Γ_2 which meet at the origin and form there an interior angle $\pi\alpha>0$. On each arc Γ_1, Γ_2 an analytic function $\phi_1(x,y)$, $\phi_2(x,y)$ is prescribed such that $\phi_1(0,0)=\phi_2(0,0)$. The problem is to study the asymptotic behavior at the origin of a solution $u(x,y)$ of the differential equation which assumes the prescribed boundary values on $\Gamma_1\cup\Gamma_2$. The main result is that the solution possesses certain asymptotic expansions at the origin. These expansions are most conveniently described in terms of the two complex variables $z=x+iy$, $z^*=x-iy$. If α is irrational the asymptotic series is a power series in the quantities z, $z^{1/\alpha}$, z^*, $z^{*1/\alpha}$. If $\alpha=p/q$, a reduced fraction, then the series is in powers of z, $z^{1/\alpha}$, $z^q\log z$, z^*, $z^{*1/\alpha}$, $z^{*q}\log z^*$. The expansions are valid in arbitrary bounded sectors of the logarithmic Riemann surfaces in the z- and z^*-planes, even if $z^*\neq\bar{z}$. They may be indefinitely term-wise differentiated. (Here the author uses an as yet unpublished result of H. Lewy to the effect that u is an analytic function of z and z^* that can be extended into these larger sectors.)

By means of earlier results of H. Lewy [Univ. Calif. Publ. Math. **1** (1950), 247–280; MR **12**, 691], Wasow [Duke Math. J. **24** (1957), 47–56; MR **18**, 568] and himself [Pacific J. Math. **7** (1957), 1437–1449; MR **20** #1765], the author transforms the problem into a similar one in which Γ_1, Γ_2 are segments of the x-axis and the boundary values are zero. With the help of the ordinary Green's function for the upper half plane, the problem is then converted into an integral equation, assuming that $z^*=\bar{z}$; this is shown to possess an analytic solution, even when the restriction $z^*=\bar{z}$ is dropped. An iterative treatment of the integral equation produces the desired asymptotic developments.

W. Wasow (Madison, Wis.)

4292:

Mizohata, Sigeru. Unicité du prolongement des solutions des équations elliptiques du quatrième ordre. Proc. Japan Acad. **34** (1958), 687–692.

Calderón [Amer. J. Math. **80** (1958), 16–36; MR **21** #3675] extended the uniqueness theorems of Holmgren and Carleman for the Cauchy problem for partial differential equations to the general case of equations of arbitrary order and arbitrary numbers of independent variables (with some exceptions) without the analyticity assumptions of Holmgren but with the definite condition that the equation possess non-multiple characteristics (apparently even in the complex domain). In this paper the author considers fourth order linear elliptic equations. By using a variant of Calderón's inequalities, he obtains, under very mild restrictions, a uniqueness theorem for the Cauchy problem for the equations under consideration; multiple characteristics (in the complex domain, of course) may occur. *C. R. DePrima* (Pasadena, Calif.)

4293:

Mizohata, Sigeru. Le problème de Cauchy pour le passé pour quelques équations paraboliques. Proc. Japan Acad. **34** (1958), 693–696.

The author again employs Calderón's methods [see review above] to prove a uniqueness theorem for the initial value problem for certain second order parabolic equations. *C. R. DePrima* (Pasadena, Calif.)

4294:

Balakrishnan, A. V. Abstract Cauchy problems of the elliptic type. Bull. Amer. Math. Soc. **64** (1958), 290–291.

An abstract Cauchy problem of the elliptic type is defined as follows. A is the infinitesimal generator of a strongly continuous 1-parameter semi-group $T(t)$, $t>0$, of endomorphisms over a Banach space X. It is required to find a function $u(t)$, $t>0$, with values in X such that: $u^k(t)$, the kth derivative of $u(t)$, is absolutely continuous for $0\leq k\leq n-1$; $u^n(t)=(-1)^{n+1}Au(t)$; and

$$\lim_{t\to 0^+}\|u^k(t)-u_k\| = 0, \quad k = 0, \cdots, n-1,$$

where $u_0, \cdots, u_{n-1}$ are prescribed elements of X. The following results are stated. The problem has at most one solution if (*) $\int_1^\infty \|T(t)\|t^{-s-1}dt<\infty$ for every $s>0$. If $n=2$ and $\limsup_{t\to\infty} t^{-1}\operatorname{Log}\|u(t)\|\leq 0$, then if $u(t)$ is a solution of the Cauchy problem satisfying (*), it must be given by

$$u(t) = (t/2\pi^{1/2})\int_0^\infty T(r)u_0 r^{-3/2}\exp(-t^2/4r)dr.$$

Applications to elliptic partial differential equations are indicated. *J. Elliott* (New York, N.Y.)

4295:

Nieto, José. Über eine Charakterisierung der elliptischen Differentialoperatoren. Arch. Math. **10** (1959), 123–125.

Ce travail est motivé par le cours de L. Schwartz [*Ecuaciones diferenciales parciales elipticas*, Univ. Nac. Colombia, Bogotá; MR **19**, 422]. Soit $\mathfrak{S}'$ l'espace des distributions tempérées φ vérifiant $(1+|\xi|^2)\hat{\varphi}\in L^2$, où $\hat{\varphi}$ est la transformée de Fourier de φ. Soit $\mathfrak{L}_\Omega{}^s$ (s réel) l'espace des distributions φ telles que $\alpha\varphi\in\mathfrak{S}'$ pour toute $\alpha\in\mathfrak{D}_\Omega$. Un opérateur différentiel à coefficients indéfiniment différentiables D (défini dans un ouvert $\Omega\subset R^N$) est dit quasi-elliptique si $\varphi\in\mathfrak{D}_\Omega{}'$ et $D\varphi\in\mathfrak{L}_\Omega{}^{s-m}$ implique toujours $\varphi\in\mathfrak{L}_\Omega{}^s$, où m est l'ordre de l'opérateur D et s est un nombre réel quelconque. L'A. énonce que l'opérateur D est quasi-elliptique si et seulement si D est elliptique. La démonstration est esquissée. *S. Mizohata* (New York, N.Y.)

4296:

Vološina, M. S. Some boundary problems for strongly elliptical systems of differential equations of the second order. Dopovidi Akad. Nauk Ukraïn. RSR **1959**, 364–368. (Ukrainian. Russian and English summaries)

The author determines types of boundary operators for which the kernel of the integral potential has no other singularities than those of the point type. He gives necessary and sufficient conditions for such operators. *H. P. Thielman* (Ames, Iowa)

4297:

Hille, Einar. Quelques remarques sur l'équation de la chaleur. Rend. Mat. e Appl. (5) **15** (1956), 102–118.

Let C_ρ, with $\rho\geq 0$, be the space of functions $f(x)$ which are continuous in $(-\infty,\infty)$ and such that finite $\lim_{x\to\pm\infty} f(x)\exp(-|x|^\rho)$ exist. It is a Banach space by the norm $\|f\|_\rho=\sup_x|f(x)|\exp(-|x|^\rho)$. Given $f(x)\in C_\rho$, the "abstract Cauchy problem" (PAC) of the heat equation (1) $\partial u/\partial t=\partial^2 u/\partial x^2$, $-\infty<x<\infty$, $t>0$, in the space C_ρ concerns the solution $u(x,t,f)$ of the equation (1) such that (i) for every $t>0$, u belongs to C_ρ and is C_ρ-strongly continuously differentiable in t, and (ii) C_ρ-strong $\lim_{t\downarrow 0} u=f$. By a detailed discussion of the classical formula

$$W(x,t,f) = \int_{-\infty}^{\infty}(4\pi t)^{-1/2}\exp(-s^2/4t)f(s+x)ds,$$

the author obtains a complete classification of the PAC according to the magnitude of ρ. (1) $0\leq\rho\leq 1$: For every $f\in C_\rho$, W gives a unique solution of PAC. (2) $1<\rho\leq 2$: If $W\in C_\rho$ for every $t>0$, then W gives a unique solution of the PAC. But there exists an $f\in C_\rho$ such that the corresponding PAC has no solutions. (3) $2<\rho$: Even if the PAC has a solution it is not unique. There exist no solutions which are positive, increasing in x and belonging to $C_\rho\ominus C_2$. *K. Yosida* (Tokyo)

4298:

Yosida, Kôsaku. An abstract analyticity in time for solutions of a diffusion equation. Proc. Japan Acad. **35** (1959), 109–113.

Let us consider the equation (*) $\partial u/\partial t=Au$, $t>0$, where

$$A = a^{ij}(x)\frac{\partial^2}{\partial x_i\partial x_j}+b^j(x)\frac{\partial}{\partial x_j}+c(x) \quad (x\in E^m)$$

is an elliptic operator and a^{ij}, $\partial a^{ij}/\partial x_k$, $\partial a^{ij}/\partial x_k\partial x_l$, b^j, $\partial b^j/\partial x_k$ and c are real functions of $C^\infty(E^m)$ bounded on E^m. The author's main results are the following: (1) For all $f\in L^2(E^m)$ there exists a solution $u(t,x)$ of (*) so that

$$\|u(t,\cdot)-f(\cdot)\|_{L^2(E^m)}\to 0 \text{ for } t\to 0;$$

(2) if for a fixed $t_0>0$, $u(t_0, x)\equiv 0$ on an open set $G_0\subset E^m$, then $u(t, x)=0$ for all $t>0$ and $x\in G_0$. The last very interesting result is deduced from the fact (established also by the author) that A (considered in a suitable manner in $L^2(E^m)$ generates an analytic semi-group of operators. *C. Foiaş* (Bucharest)

4299:

Montaldo, Oscar. Un problema di valori al contorno per un'equazione parabolica del quarto ordine. Rend. Sem. Fac. Sci. Univ. Cagliari **28** (1958), 1–8.

Consider the equation (1) $\partial^4u/\partial x^4-2\lambda\partial^3u/\partial x^2\partial y+\partial^2u/\partial y^2=0$ in the domain $\chi_1(y)<x<\chi_2(y)$, $0<y<h$, and the boundary conditions (2) $a_i(y)u+b_i(y)u_x+c_i(y)u_{xy}=f_i(y)$ and $\alpha_i(y)u_x+\beta_i(y)u_{xx}+\gamma_i(u_{xxx}=\varphi_i(y)$ on $x=\chi_i(y)$ $(i=1, 2)$, (3) $a(x)u+b(x)u_y=f(x)$ and $\alpha(x)u_y+\beta(x)u_{yy}=\varphi(x)$ on $y=0$. Under some continuity assumptions the author proves the existence of a solution of (1), (2), (3) for any positive λ. The method involves generalized potential theory for parabolic equations.

A. Friedman (Minneapolis, Minn.)

4300:

Pogorzelski, W. Étude de la matrice des solutions fondamentales du système parabolique d'équations aux dérivées partielles. Ricerche Mat. **7** (1958), 153–185.

This paper contains proofs of results previously announced [Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 79–83; MR **21** #2123].

J. Elliott (New York, N.Y.)

4301:

Pogorzelski, W. Propriétés des solutions du système parabolique d'équations aux dérivées partielles. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 369–373.

This note contains statements of theorems proved in the paper reviewed below. *J. Elliott* (New York, N.Y.)

4302:

Pogorzelski, W. Propriétés des solutions du système parabolique d'équations aux dérivées partielles. Math. Scand. **6** (1958), 237–262.

This is a continuation of the author's study [#4300] of parabolic systems of partial differential equations. Volume potentials associated with such systems are defined and their properties studied. In the last section the results are applied to the solution of the Cauchy problem for parabolic systems.

J. Elliott (New York, N.Y.)

4303:

Dobrescu-Purice, Lucia. Propagation des perturbations dans le cas des équations du 2e ordre, du type hyperbolique. Com. Acad. R. P. Romîne **8** (1958), 125–131. (Romanian. Russian and French summaries)

The author studies the geometric theory of propagation phenomena defined by hyperbolic equations, basing herself on previous work by N. Teodorescu [Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **5** (1953), 19–39; MR **17**, 101]. *J. L. Massera* (Montevideo)

4304:

Yamaguti, Masaya. Sur l'inégalité d'énergie pour le système hyperbolique. Proc. Japan Acad. **35** (1959), 37–41.

Employing the singular integral operators of Calderón and Zygmund [Amer. J. Math. **79** (1957), 901–921; MR **20** #7196] and Calderón [ibid. **80** (1958), 16–36; MR **21** #3675], the author derives an a-priori estimate for certain hyperbolic first order systems of partial differential equations. Let B, A^k, $k=1, \cdots, n$, be $N\times N$ matrices whose entries $a_{ij}^{(k)}(x, t)$ are bounded C^∞ functions on $R^n\times T$, where T is the non-negative real t-axis; $u(x, t)$, $f(x, t)$ are functions from $R^n\times T$ to R^N; then the system can be written

$$(1)\qquad \frac{\partial u}{\partial t}=\sum_{k=1}^{n}A^k\frac{\partial u}{\partial x_k}+Bu+f.$$

The estimate which is derived is the usual estimate; i.e., if $u(\cdot, t)\in L_2(R^n)$, has a strong gradient in $L_2(R^n)$, is strongly differentiable in t and satisfies (1) with $f(\cdot, t)\in L_2(R^n)$ and strongly integrable in t, $0\le t\le h$,

$$(2)\qquad \|u(\cdot, t)\|^2\le C_1e^{ct}\{\|u(\cdot, 0)\|^2+\int_0^t\|f(\cdot, t)\|^2dt\},$$

where C_1, c depend only on A^k, B, and h.

The special assumptions placed upon (1) in order to prove (2) are that the characteristic matrix $\lambda I-\sum_{k=1}^{n}A^k\xi_k$, ξ_k real, consists of r square blocks along the main diagonal, all other entries being zero; the characteristic roots $(\sum_{k=1}^{n}\xi_k{}^2=1)$ for each block are real, distinct, and bounded away from each other.

C. R. DePrima (Pasadena, Calif.)

4305:

Gusarov, L. A. The Cauchy problem for certain systems of linear partial differential equations. Dokl. Akad. Nauk SSSR **125** (1959), 966–969. (Russian)

The author considers the Cauchy problem

$$u_t-P(t, \partial/\partial x)u+f(t, x)=0,\quad u(x, t_0)=u^*(x).$$

Here P is an m by m matrix whose elements are linear operators of various orders with complex coefficients which are continuous in t; $u=(u_1, \cdots, u_m)$, $x\in R_n$, $t_0\le t\le T$. The functions f and u^* are assumed to be bounded continuous complex-valued, and to be of class $C^{(2n+1)\nu}$ in x, for some ν too complicated to define here. Under a large number of involved assumptions on the coefficients in the equation it is asserted that the problem is uniformly correctly set on $t_0\le t\le T$ in the class of bounded functions. Reference should be made to the earlier work of Gel'fand and Šilov [Uspehi Mat. Nauk (N.S.) **8** (1953), no. 6 (58), 3–54; MR **15**, 867] referred to in the paper.

E. A. Coddington (Los Angeles, Calif.)

4306:

★Bochner, S. Systèmes des équations aux dérivées partielles qui se rattachent aux formules de Green. La théorie des équations aux dérivées partielles. Nancy, 9–15 avril 1956, pp. 19–27. Colloques Internationaux du Centre National de la Recherche Scientifique, LXXI. Centre National de la Recherche Scientifique, Paris, 1956. 187 pp. 1500 francs.

Let

$$\Omega[f(\xi);\xi-x]=\sum\Omega_{\alpha_1\cdots\alpha_{n-1}}d\xi_{\alpha_1}\wedge d\xi_{\alpha_2}\wedge\cdots\wedge d\xi_{\alpha_{n-1}}$$

be an exterior differential form of dimension $n-1$ in the Euclidean space E_n. Here $f(\xi)\equiv f(\xi_1, \cdots, \xi_n)$ and each Ω-term has the form

$$\sum G_{p_1\cdots p_n}(\xi-x)D^{p_1+\cdots+p_n}f(\xi).$$

The G-coefficients, in turn, are defined for $x\neq\xi$ and depend only on the difference vector $\xi-x$. All functions considered are of class C^∞, wherever defined. The exterior derivative of Ω leads to the differential operator $\sum G_j(\xi-x)T_jf(\xi)$ $(j=1, \cdots, k)$, where the G_j are again functions of $x-\xi$ and each T_j is a differential operator with constant coefficients of the form

$$\sum c_{q_1\cdots q_n}D^{q_1+q_2+\cdots+q_n}f(\xi) = Tf(\xi).$$

The author studies now the set of differential equations (*) $T_jf(x)=\rho_j(x)$ under the obvious integrability conditions $T_j\rho_l=T_l\rho_j$. Let B_n be a bounded domain in E_n with sufficiently regular boundary B_{n-1}; define the linear operator

$$Lf(x) = \int_{B_{n-1}}\Omega[f(\xi);\,\xi-x]-\int_{B_n}\sum G_j(\xi-x)T_jf(\xi)d\xi.$$

It is easy to show that if one can find a function $f(x)$ which satisfies (*) in the neighborhood of B_{n-1} then one can construct a function $g(x)$ which satisfies $T_jg=L\rho_j$ in the entire domain B_n. The most important differential operators lead to the particular linear operator $Lg\equiv g$, and in this case the knowledge of a solution f of (*) near the boundary guarantees the existence of a (possibly different) solution g of the same problem in the entire domain. The question of unique continuation is discussed for solutions $f(x)$, defined near the boundary. The main result of the paper is as follows. Hypothesis H: We suppose that the space E_n is a product of Euclidean subspaces $E=E_{n_1}\times E_{n_2}\times\cdots\times E_{n_k}$ such that if the function $\varphi(x)$ depends only on the coordinates of E_{n_j} one has $T_l\varphi=0$ for $l\neq j$. Theorem: "Let D be a bounded product domain $D=D_1\times D_2\times\cdots\times D_k$ with $D_j\in E_{n_j}$ and suppose the validity of hypothesis H. If the ρ_j satisfy the integrability conditions, there exists a function $f(x)$ which satisfies in D the differential system (*)." This result generalizes considerably previous existence theorems of the author which are of significance in the theory of analytic functions of several complex variables [Ann. of Math. (2) 44 (1943), 652–673; MR 5, 116; in particular pp. 664–665]. *M. Schiffer* (Stanford, Calif.)

POTENTIAL THEORY

See also 4581a–b.

4307:

Samara, H. Transformations of the Laplace equation. Proc. Iraqi Sci. Soc. **2** (1958), 7–10. (Arabic summary)

It is shown that the cylindrical polar coordinate transformation is the only one which transforms the Laplace equation in three variables into the form

$$(*)\qquad u_1^{-1}(\partial/\partial u_1)(u_1\partial\varphi/\partial u_1)+\partial^2\varphi/\partial u_2^2 = 0,$$

under the assumption that φ is independent of u_3, the third variable of the orthogonal curvilinear coordinate system (u_1, u_2, u_3). The author states that the same analysis applies in the case where (*) is replaced by

$$u_1^{-1}(\partial/\partial u_1)(u_1\partial\varphi/\partial u_1)+\partial^2\varphi/\partial u_2^2+\alpha(\partial/\partial u_3)(\beta\partial\varphi/\partial u_3) = 0$$

for arbitrary α, β. In that case the only transformation is again the cylindrical polar system and $\alpha=u_1^{-2}$, $\beta=1$.

V. G. Hart (Cork)

4308:

Martin, M. H. Linear and nonlinear boundary problems for harmonic functions. Proc. Amer. Math. Soc. **10** (1959), 258–266.

Cet article concerne la construction d'identités intégrales permettant d'étudier l'unicité de la fonction u, harmonique dans une région S du plan, et qui, le long de la frontière C de S, vérifie une condition de la forme

$$u_n = h(s)f(u)$$

où u_n désigne la dérivée normale extérieure. L'auteur établit l'identité

$$\int_C\lambda\left(u_2\frac{\partial u_1}{\partial n}-u_1\frac{\partial u_2}{\partial n}\right)ds =$$

$$\int_S[u_2^2(\lambda_x^2+\lambda_y^2)+\lambda(u_2\Delta u_1-u_1\Delta u_2)]dS,\quad \lambda = u_1/u_2,$$

valable pour u_1 et u_2 de classe $C^{(2)}$ dont le rapport λ est une fonction régulière. Il s'ensuit un théorème d'unicité pour le problème de Steklov qui correspond à $f=u$.

Comme exemple de problème non linéaire, le cas où $f=u^{1+p}$ (p entier, pair) est considéré. Une identité intégrale obtenue par l'Auteur permet de conclure que si u_2 est une solution, non constante, il ne peut exister une autre solution u_1 pour laquelle $\lambda=u_1/u_2$ est régulier et $|\lambda|<1$. *R. Gerber* (Grenoble)

4309:

Ohtsuka, Makoto. Capacité des ensembles produits. Nagoya Math. J. **12** (1957), 95–130.

Après quelques rappels concernant la théorie des potentiels d'ordre α sur un espace métrique localement compact, l'auteur cherche des minorations et des majorations pour la capacité d'un ensemble produit. Exemple de minoration:

$$C_i^{(\alpha+\beta)}(X\times Y)\geq\alpha^{-\alpha/2}\beta^{-\beta/2}(\alpha+\beta)^{(\alpha+\beta)/2}C_i^{(\alpha)}(X)C_i^{(\beta)}(Y)\qquad(\alpha\text{ et }\beta>0)$$

où $C_i^{(\gamma)}(Z)$ désigne la capacité intérieure d'ordre γ de l'ensemble Z. Cette inégalité s'inspire d'un résultat analogue démontré par Marstrand [Proc. Cambridge Philos. Soc. **50** (1954), 198–202; MR **15**, 691] pour les mesures hausdorffiennes de deux ensembles de la droite réelle.

Dans le cas où X est un ensemble borné de R^n on donne une majoration de $C_i^{(\alpha+\beta)}(X\times Y)$ en fonction de $C_i^{(\alpha)}(X)$, majoration d'où il résulte que, lorsque X et Y sont des compacts de R^n et R^m, on a

$$\dim(X\times Y)\leq\inf\{n+\dim(Y);\,m+\dim(X)\},$$

où $\dim(Z)$ représente la dimension capacitaire du compact Z, ou, ce qui revient au même, sa dimension hausdorffienne. Ceci généralise un résultat donné par Besicovitch et Moran [J. London Math. Soc. **20** (1945), 110–120; MR 8, 18] pour $m=n=1$, et on construit un exemple pour lequel l'égalité a lieu. *J. Deny* (Palaiseau)

FINITE DIFFERENCES AND FUNCTIONAL EQUATIONS

See also 4198, 4210.

4310:

Mitrinovitch, Dragoslav S. Nouvelles formules relatives aux nombres de Stirling. C. R. Acad. Sci. Paris **248** (1959), 1754–1756.

Put $\prod_{r=0}^{n-1}(x-r)=\sum_{m=1}^{n}S_n^m x^m$; then S_n^m satisfies

$$(*)\qquad S_{n+1}^m = S_n^{m-1} - nS_n^m.$$

The author remarks that the general solution of (*) is unknown. He seeks solutions of the form

$$S_n^{n-k}=\binom{n}{k+1}(A_1^{(k)}n^{k-1}+\cdots+A_k^{(k)}).$$

In a previous paper [Acad. Roy. Belgique. Bull. Cl. Sci. (5) **33** (1947), 244–247; MR **9**, 139] he had found such solutions for $k=2, 3, 4, 5$. He now computes the solutions for $k=6, 7, 8$ and remarks that one can continue the process for larger values of k. *L. Carlitz* (Durham, N.C.)

4311:

Rosenbaum, R. A. An application of matrices to linear recursion relations. Amer. Math. Monthly **66** (1959), 792–793.

4312:

Eliáš, Jozef. Über eine Operatorenmethode bei den Differenzengleichungen. Mat.-Fyz. Časopis. Slovensk. Akad. Vied **8** (1958), 203–227. (Slovak. Russian and German summaries)

The author develops an analogue of Mikusiński's theory of convolution quotients [Mikusiński, *Rachunek operatorów*, Polskie Towarzystwo Matematyczne, Warsaw, 1953; MR **16**, 243; see #4333 below] for functions of a non-negative integer, i.e., sequences. a or $\{a(n)\}$ denotes such a function and $a(n)$ its value at the integer n, where $n=0, 1, 2, \cdots$. Addition of two such functions, and multiplication of a function by a number are defined in the obvious way, and the convolution of two sequences a and b is the sequence ab defined by $ab(0)=0$, $ab(n)=\sum_{k=1}^{n}a(n-k)b(k-1)$, $n=1, 2, \cdots$. The set of sequences endowed with the operations of addition and convolution forms a commutative ring. In this ring there are no divisors of zero, and thus the ring can be embedded in a quotient field, the field of convolution quotients. The function $l=\{1\}$ is the operator of summation, and its reciprocal, s, in the field is the operator of extended forward differentiation,

$$s\{a(n)\}=\{a(n+1)-a(n)\}+a(0).$$

Rational functions of s are next interpreted and then the results are applied to the solution of linear difference equations. {Reviewer's remarks: (i) The method developed here is closely related to the well-known technique of using formal generating functions for the solution of difference equations. (ii) Mikusiński's operational calculus can also be used to solve difference equations; see Part II, Chapter III of the 1957 edition of Mikusiński's book.}

A. Erdélyi (Pasadena, Calif.)

4313:

Ganapathy Iyer, V. On permutable integral functions. J. London Math. Soc. **34** (1959), 141–144.

Two functions $f(z)$ and $g(z)$ are said to be permutable [cf. G. Julia, C. R. Acad. Sci. Paris **173** (1921), 690–693] if $f[g(z)]=g[f(z)]$. After theorem 1, enumerating some functions obviously permutable with a given $f(z)$, theorem 2 states that if $f(z)$ is a polynomial of degree greater than 1 then the function $g(z)$, if integral, is also a polynomial. Theorem 3, resp. 4, gives for $f(z)=az^m$ $(m>1)$, resp. $f(z)=az+b$, all permutable integral functions g. In a remark due to the referee, it is mentioned that $f(z)=z$ is the only integral function permutable with all integral functions. {Reviewer's remarks: Since under certain conditions even all permutable rational functions are known [J. Ritt, Trans. Amer. Math. Soc. **25** (1923), 399–448; cf. for polynomials G. Julia, op. cit., and E. Jacobsthal, Math. Z. **63** (1955), 243–276; MR **17**, 574], theorem 3 must follow from theorem 2. On the other hand, the author extends his theorem 3 also to arbitrary functions regular at $z=0$ in place of integral ones. In connection with theorem 4, the complete solution of the functional equation $g(az+b)=cg(z)+d$ [J. Aczél, Mat. Lapok **5** (1954), 48; A. Császár, ibid. **6** (1955), 353–355; J. Aczél, ibid. **7** (1956), 127, 144] might be mentioned. A minor misprint: p. 143, l. 15, read $d(a-1)$ instead of $a(d-1)$.}

J. Aczél (Debrecen)

4314:

Hosszú, Miklós. A generalization of the functional equation of distributivity. Acta Sci. Math. Szeged **20** (1959), 67–80.

The author obtains general solutions of various functional equations; the functions are defined on domains in Euclidean space and subject to differentiability conditions (whether continuity alone suffices is presumably not known).

This theorem, at dimension one, is a more specific indication of the questions considered. Let I be an open interval and $M: I\times I\to I$ an idempotent algebra having a continuous mixed partial derivative at all points of the diagonal; furthermore let $N: I\times U\to I$ describe a transitive system of endomorphisms of M (including the identity) where U is (say) another interval. Then M is topologically isomorphic to a $\Phi: I'\times I'\to I'$ having affinities as operators.

Relations to Cauchy's equation, homogeneous functions, isotopies are discussed. *S. Stein* (Davis, Calif.)

SEQUENCES, SERIES, SUMMABILITY

See also 4175, 4213.

4315:

Šalát, Tibor. Über gewisse Räume der Reihen mit Baireschen Metrik. Mat.-Fyz. Časopis. Slovensk. Akad. Vied **7** (1957), 193–206. (Slovak. Russian and German summaries)

The author continues earlier investigations [same Časopis **4** (1954), 203–211; **5** (1955), 94–100; Acta Fac. Nat. Univ. Comenian. Math. **2** (1957), 71–76; MR **16**, 1099; **17**, 728; **20** #2554]. He considers the set X of all series x of type $\sum \varepsilon_i a_i$, where $\varepsilon_i=1$ or -1 $(i=1, 2, \cdots)$

and $\sum a_i$ is a fixed convergent series of positive terms. By $S(x)$ he denotes the sum of the series x, by W the set of all $S(x)$, by R_k the remainders of $\sum a_k$, hence $R_k = a_{k+1} + \cdots$. In X a distance ρ of standard type is introduced, which makes X a compact space and $S(x)$ a continuous function. If $a_k = R_k$ for at least one k or if $a_k \leqq R_k$ for $k > k_0$, then $S(x)$ is not one-to-one; if however (*) $a_k > R_k$ for all k, then $S(x)$ is a homeomorphism (th. 1). If (*) holds, then the geometric structure of W shows that fixing of one coordinate ε_i halves the measure of W (th. 2 and 3). If (*) holds and if $f(n, x)$ denotes the quantity of integers 1 in $\varepsilon_1, \cdots, \varepsilon_n$, then for almost all $w \in W$ the corresponding x satisfies

$$\liminf_n n^{-1} f(n, x) \leqq 1/2 \leqq \limsup_n n^{-1} f(n, x)$$

(th. 4; proof by counting combinations of the ε_i). If, contrary to the previous assumption, $\sum a_i = +\infty$, then the set A of series x whose partial sums are bounded from above is—as one would expect—a G_δ-set of first category in (X, ρ). *K. Zeller* (Tübingen)

4316:

Buschman, R. G.; and Durbin, J. R. Integral transformations and closed-form expressions for sums of infinite series. Amer. Math. Monthly **66** (1959), 789–790.

The authors show that if the terms of the series $\sum a_n g(n+1)$ can be expressed as products of the coefficients of a known power series $\sum a_n x^n$ and of the Mellin transform of a function f, then the series can be expressed as the definite integral $\int_0^\infty \{\sum a_n x^n\} f(x) dx$. Standard tables of the appropriate integral transform may then be used to find the sum of the series. An illustration of this technique is also given.

M. S. Ramanujan (Ann Arbor, Mich.)

4317:

Parameswaran, M. R. Some applications of functional analysis in summability. Math. Student **26** (1958), 93–104.

The author gives a survey of certain functional analytic techniques used for the investigation of summability methods, mainly of the type A-$\lim x_k = \lim_n \sum_k a_{nk} x_k$. The norm $\|A\| = \sup_n \sum_k |a_{nk}|$ is helpful in proving Mercerian theorems (Agnew, Parameswaran) and in determining bounded summability fields (Ganapathy Iyer). The summability field $\mathfrak{S}_C A$ of A can be considered as an F-space (Mazur and Orlicz; $\mathfrak{S}_C$ denotes the set of convergent sequences). The category principle then yields "inequivalence theorems", for instance for the C_∞-method. Another useful concept is the perfect part of a regular A, i.e., the closure of $\mathfrak{S}_C$ in $\mathfrak{S}_C A$. Every bounded A-summable sequence lies in the perfect part. As a consequence we have the following theorem. If a conservative A sums a divergent sequence, then it sums an unbounded sequence. On the other hand, a conservative A sums a bounded divergent sequence if and only if its perfect part is $\neq \mathfrak{S}_C$ (Wilansky). This enables us to deduce unrestricted gap theorems (like the High Indices Theorem due to Hardy and Littlewood) from gap theorems for bounded sequences (Meyer-König). For many methods "best" Tauberian gap theorems are known. The author extends these results by comparing methods like C^n and B under gap conditions [J. Indian Math. Soc. (N.S.) **22** (1958), 85–92]. Other applications concern "Abschnittskonvergenz" (Wilansky, Zeller) and "continuous" methods (Włodarski). *K. Zeller* (Tübingen)

4318:

Jakimovski, A.; and Parameswaran, M. R. Generalized Tauberian theorems for summability-(A). Quart. J. Math. Oxford Ser. (2) **9** (1958), 290–298.

The authors give new proofs and slight generalizations of some of the Tauberian theorems obtained by Jakimovski [Pacific J. Math. **7** (1957), 943–954; MR **19**, 544] which were also proved by Rajagopal [ibid. **7** (1957), 955–960, 1727; MR **19**, 544, 1174]. Their theorems involve the Hölder transform $\{H^n s_n\}$ of the sequence $\{s_n\}$ which is defined by setting $\Delta^n t_0 = (n+1)^{-n} \Delta^n s_0$, $n = 0, 1, 2, \cdots$. The simplest theorem in the present paper is the following. Let the real sequence $\{s_n\}$ be Abel limitable to L and suppose that there is a number γ such that the sequence $\{H^{-1} s_n - \gamma s_n\}$ is bounded above. Then $s_n \to L$.

J. Korevaar (Madison, Wis.)

4319:

Roberts, J. B. Analytic continuation of meromorphic functions in valued fields. Pacific J. Math. **9** (1959), 183–193.

Analytic continuation of power series by certain types of matrix methods in arbitrary fields k complete with respect to a valuation is studied in this paper. The methods are applicable in Archimedean or in non-Archimedean fields. The following notations and definitions are employed. The set of all infinite series with terms in k is denoted by S; the subset of S consisting of all unconditionally convergent series is denoted by T, and the set of series in T which converge to non-zero limits is denoted by T^*. If $C = \sum_{i=0}^\infty c_i$ is in S, then the matrices used are (a) $B_C = (b_{i,j})$, where $b_{i,j} = c_{j-i}$, and c_{j-i} is taken to be 0 when $j < i$; (b) if C converges to $c \neq 0$, then $A_C = B_C(c)^{-1}$, where (c) is the diagonal matrix with all diagonal elements c.

The weak topology in M, the set of all infinite matrices $(a_{i,j})$ with $a_{i,j}$ in k for all i, j, is the topology induced on M by making the sequence $m_n = (a_{i,j}^{(n)})$ of matrices converge to the matrix $(a_{i,j})$ if and only if for all i, j we have $a_{i,j}^{(n)} \to a_{i,j}$; $\{m_n\}$ is then said to converge weakly to $(a_{i,j})$. If, for an arbitrary positive real number r, we denote the set of all matrices $(a_{i,j})$ with $|a_{i,j}| < r$ for all i, j by M_r, this induces on M the topology of the additive group of M, and is called the uniform topology.

The following results are established. Theorem 1: Let $\sum_{n=1}^\infty C_n$ converge, and suppose C_n is in T^* for all n. Then $\prod_{n=1}^\infty A_{C_n}$ converges relative to both weak and uniform topologies of M, and its limit is $A_{\prod_{n=1}^\infty C_n}$. Theorem 2: For $x_0 \neq 0$, and for $n = 0, 1, 2, \cdots$, let $C_n(x_0) = \sum_{i=0}^\infty a_{n,i} x_0^i$ and $C(x_0) = \sum_{i=0}^\infty c_i(x_0)$ be in T. Then if $C_n(x_0) \to C(x_0)$, we have (i) for each i there is an a_i such that $\lim_{n\to\infty} a_{n,i} = a_i$; (ii) $a_i x_0^i = c_i(x_0)$; i.e., $C(x_0)$ is the term by term limit of $C_n(x_0)$ when it is the limit in the T norm. Theorem 3: Let $C(x)$ be an entire power series. Then for x a non-zero of the sum function $c(x)$ of $C(x)$ we have

$$A_{C(x)} = A_{a_0} A_{x^{i_0}} \prod_{q=0}^{n \leqq \infty} A_{1 - x/z_q},$$

where, if n is infinite, the right-hand side converges to the left in both the uniform and weak topologies of M; i_0 is

the multiplicity of the zero $x=0$ of $c(x)$, and z_q ranges over the set of zeros of $c(x)$, each factor $1-x/z_q$ occurring a number of times equal to the multiplicity of z_q as a zero of $c(x)$. $[f(x)]_\alpha$ denotes the Taylor series $\sum_{i=0}^{\infty} a_i(x-\alpha)^i$ about α in k, which converges to $f(x)$ in its circle of convergence. Two theorems are given concerning continuation of power series $[f(x)]_\alpha$ when $f(x)$ is meromorphic on a circle D in k. P. Vermes [Proc. Edinburgh Math. Soc. (2) 8 (1947), 1–13; 8 (1948), 43–49; Amer. J. Math. **71** (1949), 541–562; MR **9**, 234; **10**, 291, 699], using series to sequence methods, has dealt with similar problems when k is the field of complex numbers; some of his results in the first of the references above overlap some of the work of the present paper. *R. G. Cooke* (London)

4320:

Tanaka, Chuji. On Julia-lines of Dirichlet series. Kōdai Math. Sem. Rep. **10** (1958), 161–171.

Let

$$(1)\qquad F(s)=\sum a_n \exp(-\lambda_n s),$$

$$s=\sigma+it,\quad 0\leqq\lambda_1<\lambda_2<\cdots,\quad \lambda_n\to\infty,$$

be simply convergent in the whole plane. If $F(s)$ takes on every value, except perhaps two (∞ included), infinitely often in the strip $|t-t_0|<\varepsilon$ for every $\varepsilon>0$, then the line $t=t_0$ is called a Julia-line. If instead $|F(s)|$ tends uniformly to infinity, and $\arg F(s)$ assumes every argument θ (mod 2π) infinitely often in the strip $|t-t_0|<\varepsilon$ for every $\varepsilon>0$, then the line $t=t_0$ is called the argument-line.

The author, generalizing theorems of S. Mandelbrojt [*Séries lacunaires*, Actualités Sci. Industr. no. 305, Hermann, Paris, 1936; *Séries adhérentes, régularisation des suites, applications*, Gauthier-Villars, Paris, 1952; MR **14**, 542], obtains the following result. Let

$$\liminf_{n\to\infty}(\lambda_{n+1}-\lambda_n)=h>0,\quad \limsup_{n\to\infty}\frac{n}{\lambda_n}=\delta\left(\leqq\frac{1}{h}\right)$$

and let (1) be simply convergent in the whole plane. Let $F(s)$ be of order ρ. If $\rho>0$ then (1) has at least one Julia-line in any strip $|t-t_0|\leqq\pi\delta$, t_0 arbitrary but fixed. If $\rho=0$, then there exists at least one Julia-line or an argument-line in any strip $|t-t_0|\leqq\pi\delta$. It is to be noticed that the width of the strip is independent of ρ. The case $\delta=0$ is of particular interest.

M. S. Robertson (New Brunswick, N.J.)

APPROXIMATIONS AND EXPANSIONS

4321:

Fernandez Avila, Francisco Javier. Generalization of vector spaces by means of the Stieltjes integral. Mem. Mat. Inst. "Jorge Juan", no. 19, 54 pp. (1957). (Spanish)

Same as Rev. Mat. Hisp.-Amer. (4) **17** (1957), 22–37, 150–160, 209–223, 278–290 [MR **19**, 956].

4322:

Lesky, P.; und Kaiser, A. Über die Interpolationspolynome von Tschebyschew. Monatsh. Math. **63** (1959), 277–286.

Tchebyschew's interpolation polynomials can be defined by orthogonalization of the sequence $1, x, x^2, \cdots, x^N$ under the inner product $(P, Q)=\sum_{n=0}^{N} P(n)Q(n)$. They can be obtained as the solution of a minimum problem, the kth vector of the orthogonalized sequence having the direction of the shortest distance from the origin to the set of all polynomials of degree k with the highest coefficient 1. The main object of the authors is to show that this minimum problem can be solved explicitly by the calculus of variations, and that the well-known formulas for these orthogonal polynomials are thus obtained in a natural way.

N. G. de Bruijn (Amsterdam)

4323:

Korneičuk, N. P. Approximation of a class of functions by Bernstein-Rogosinski sums. Dopovidi Akad. Nauk Ukraïn. RSR **1959**, 359–363. (Ukrainian. Russian and English summaries)

The author indicates proofs for some properties of

$$E_n^{(m)}(\alpha)=\sup_x \max |B_n^{(m)}(f,x)-f(x)|$$

$$(m=1,2,\cdots,2n-1),$$

where $B_n^{(m)}$ is obtained from the partial sums $S_n(x)$ of the Fourier series of $f(x)$ by the formulas

$$B_n^{(0)}(x)=S_n(x),\quad B_n^{(k)}(x)=\tfrac{1}{2}B_n^{(k-1)}(x+\pi(2n+1)^{-1})+\tfrac{1}{2}B_n^{(k-1)}(x-\pi(2n+1)^{-1}),$$

and the supremum is taken for all functions satisfying the Lipschitz condition $|f(x_1)-f(x_2)|\leqq|x_1-x_2|^\alpha$. For example, $E_n^{(m)}(1)<E_n^{(m+1)}(1)$, $m\leqq 2n-2$. Compare also his previous paper [Dokl. Akad. Nauk SSSR **125** (1959), 258–261; MR **21** #793]. *G. G. Lorentz* (Syracuse, N.Y.)

4324:

Weiss, G. H.; and Maradudin, A. A. On an asymptotic expansion occurring in the evaluation of a lattice sum. Canad. J. Phys. **37** (1959), 965–966.

It is shown how an expansion of the function

$$F(t)=\sum_{n=0}^{\infty} n\exp(-n^2t)$$

for small values of t may be obtained using Euler's summation formula. Such an expansion is required for the evaluation of certain lattice sums.

A. C. Hurley (Melbourne)

FOURIER ANALYSIS

See also 4209, 4323, 4334, 4561.

4325:

★**Byerly, William Elwood. An elementary treatise on Fourier's series and spherical, cylindrical, and ellipsoidal harmonics, with applications to problems in mathematical physics.** Dover Publications, Inc., New York, 1959. ix+287 pp. Paperbound: $1.75.

Unaltered republication of the last edition [Ginn, Boston, 1902].

4326:

Tandori, Károly. Bemerkung zur Divergenz der trigonometrischen Reihen. Acta Sci. Math. Szeged **20** (1959), 25–32.

The author gives another proof of a theorem of Stečkin [Izv. Akad. Nauk SSSR. Ser. Mat. **21** (1957), 711–728; MR **20** #3412] that given a monotonic non-increasing sequence $\{\rho_n\}$ of positive real numbers such that $\sum \rho_n^2 = \infty$, there exists a power series $\Sigma = \sum a_n z^n$, with $|a_n| = \rho_n$, such that both the real part and the imaginary part of Σ are everywhere divergent on the unit circle.

L. A. Rubel (Urbana, Ill.)

4327:

Yano, Kenji. On convergence criteria for Fourier series. II. Tôhoku Math. J. (2) **11** (1959), 54–62.

[For part I, see Proc. Japan Acad. **34** (1958), 331–336; MR **20** #3413.] "Let $\phi(t)$ be integrable in $(0, \pi)$, even, periodic of period 2π. Let $\phi(t) \sim a_0/2 + \sum a_n \cos nt$, $\Phi_\alpha(t) = (1/\Gamma(\alpha)) \int_0^t (t-u)^{\alpha-1}[\phi(u) - s]du$, and $s_n = a_0/2 + \sum_1^n a_\nu$. F. T. Wang [Proc. London Math. Soc. **17** (1942), 98–107; MR **4**, 37] has proved that if $\beta > \alpha > 0$, $\phi(t) \in L$, $\Phi_\alpha(t) = o(t^\beta)$ as $t \to 0$ and if $a_n > -An^{-\alpha/\beta}$, $A > 0$, then $s_n \to s$ as $n \to \infty$. In order to prove this, Wang used the method of Riesz summability. In this paper the author gives an alternate proof by a method of generalized de la Vallée Poussin summability." (From the introduction)

A. Shields (New York, N.Y.)

4328:

Tureckiĭ, A. H. Saturation classes in the space C. Dokl. Akad. Nauk SSSR **126** (1959), 30–32. (Russian)

Let $\gamma_k(\xi)$ $(k = 1, 2, \cdots)$ be functions defined on a set with a limit point ω. If f is continuous of period 2π with Fourier series (1) $a_0/2 + \sum (a_k \cos kx + b_k \sin kx)$, we consider the new series (2) $a_0/2 + \sum \gamma_k(\xi)(a_k \cos kx + b_k \sin kx)$. We assume (2) is uniformly convergent for all $f \in C$ and all ξ near ω. Let $F(x, \xi)$ be the sum of (2). Let $\phi(\xi)$ be positive, monotone decreasing to zero and such that (3) $\max |f(x) - F(x, \xi)| < a\phi(\xi)$ for all $f \in C$ (a is a constant depending on f), while there exists an $f \in C$ and a constant b for which (4) $\max |f(x) - F(x, \xi)| > b\phi(\xi)$. Under these assumptions the method of summation γ is said to be saturated with degree ϕ, and the saturation class is the set of those $f \in C$ for which (4) holds.

The author states without proof the following general result, generalizing his previous work [Dokl. Akad. Nauk SSSR **121** (1958), 980–983; MR **21** #1484]. If for some integer p and constants $a_0 \neq 0, a_1, a_2, \cdots, a_p$ we have for each k the asymptotic relation $1 - \gamma_k(\xi) \sim (a_0 k^p + \cdots + a_{p-1}k + a_p)\phi(\xi)$, and if (3) holds, then (5) $|\Delta_h^p \tilde{f}| = O(h^p)$ for p even, and $|\Delta_h^p f| = O(h^p)$ for p odd. Here $\tilde{f}$ denotes the conjugate function to f, and Δ_h^p denotes the finite difference of order p. If there exists a constant K such that $\|F\| \leq K\|f\|$, then conversely (3) follows from (5).

A. Shields (New York, N.Y.)

4329:

Goldberg, Richard R. Certain operators and Fourier transforms on L^2. Proc. Amer. Math. Soc. **10** (1959), 385–390.

The author proves the following theorem. Let ψ be non-negative with $\int_0^\infty \psi(y) y^{-1/2} dy < \infty$. Then the transformation $T: g \to G$ defined by $G(x) = (1/x) \int_0^\infty \psi(y/x) g(y) dy$ is a bounded linear operator on $L_2(0, \infty)$ with the property $TU = UT^*$, where T^* is the adjoint of T, and U is the cosine transform. This generalizes a theorem of Titchmarsh [*Introduction to the theory of Fourier integrals*, Clarendon Press, Oxford, 1937; p. 93] in which ψ is the characteristic function of the interval $(0, 1)$.

K. Chandrasekharan (Bombay)

4330:

Bourion, Georges. Sur les cas classiques de convergence des séries de Fourier associées aux fonctions presque-périodiques de Bohr. Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) **2** (50) (1958), 1–4.

The title refers to two cases considered by Bohr at an early stage of the theory of almost periodic functions: Fourier series of almost periodic functions with positive coefficients or with linearly independent exponents. The note contains nothing very new. In view of the chosen method of proof it would have been natural to quote Bochner for having deduced the two theorems as corollaries of his summation theorem.

E. Følner (Copenhagen)

4331:

Stein, Elias M. Localization and summability of multiple Fourier series. Acta Math. **100** (1958), 93–147.

This paper contains results on the spherical summability of multiple Fourier series. It is an appropriate adjunct to the fundamental paper by S. Bochner [Trans. Amer. Math. Soc. **40** (1936), 175–207] and settles some of the problems that have remained open since, an account of which has been given, for instance, by the reviewer [Proc. Internat. Congress Mathematicians, 1954, Amsterdam, Vol. III, pp. 85–91, Noordhoff, Groningen, 1956; MR **19**, 268].

Let $f(x) \equiv f(x_1, \cdots, x_k)$ be a Lebesgue integrable function defined in the fundamental cube $Q_k: -\pi < x_i \leq \pi$ $(i = 1, \cdots, k)$ in Euclidean k-space. Let us write the Fourier series of f as usual:

$$f(x) \sim \sum a_n e^{i(n \cdot x)} \equiv \sum \cdots \sum a_{n_1 \cdots n_k} e^{i(n \cdot x)},$$

where $n = (n_1, \cdots, n_k)$, n_k is integral, and $n \cdot x = n_1 x_1 + \cdots + n_k x_k$. Let the spherical Riesz mean of order δ of the multiple Fourier series of f at the point x be denoted by

$$S_R^\delta(x) \equiv S_R^\delta(x, f) = \sum_{|n| < R} \left(1 - \frac{|n|^2}{R^2}\right) a_n e^{i(n \cdot x)},$$

where $|n| = (n_1^2 + \cdots + n_k^2)^{1/2}$. Then Bochner has proved [loc. cit.] that for functions $f \in L_1(Q_k)$, we have

$$\lim_{R \to \infty} S_R^\delta(x, f) = f(x) \tag{1}$$

at every point of continuity of f, provided that $\delta > \frac{1}{2}(k-1)$; and, more generally, that the validity of (1) at a given point x depends, for $\delta > \frac{1}{2}(k-1)$, only on the "local" behaviour of the function f, i.e., on the values of f in any neighbourhood of the point x. This is not in general true for values of δ less than or equal to $\frac{1}{2}(k-1)$. In fact, Bochner has shown that for $k \geq 2$ there exists an $f \in L_1(Q_k)$ which vanishes in a neighbourhood of the origin for which $\limsup_{R \to \infty} S_R^\alpha(0, f) = +\infty$, where $\alpha = \frac{1}{2}(k-1)$. For this reason α is termed the "critical exponent". If, on the other hand, one restricts oneself to functions $f \in L_2(Q_k)$, then, by another result of Bochner's, the validity of (1) for the critical exponent $\delta = \alpha$ again depends only on the local behaviour of f. It has remained an open question whether localization holds for the critical exponent $\delta = \alpha$ in the case of functions belonging to $L_p(Q_k)$, $1 < p < 2$. The author answers this in the affirmative for the class of functions f for which $\int_{Q_k} |f(x)| \log^+ |f(x)| dx < \infty$, which

includes every $L_p(Q_k)$ class for $p>1$. He also obtains a series of results on the validity of (1) for almost every x, and on associated problems connected with dominated summability. He proves that if $f(x)\in L_p(Q_k)$, $p>1$, then (1) holds almost everywhere for some δ below the critical exponent, in fact for $\delta>\alpha(2/p-1)$, so that the notion of critical exponent turns out to be significant only for pointwise summability. A companion theorem is that if $\int_{Q_k}|f(x)|(\log^+|f(x)|)^2dx<\infty$, then (1) holds for $\delta=\alpha$ almost everywhere. With the notation $S_*^\delta=S_*^\delta(x,f)=\sup_{0<R<\infty}|S_R^\delta(x,f)|$, the author proves the inequality

$$\|S_*^\delta(x,f)\|_p \leq A_{p,\delta}\|f(x)\|_p \tag{2}$$

for $f\in L_p(Q_k)$, $1<p\leq 2$, and $\delta>\alpha(2/p-1)$. This is a generalization to several variables of an inequality of Hardy and Littlewood, and yields a result on norm convergence, namely

$$\lim_{R\to\infty}\|S_R^\delta(x,f)-f(x)\|_p = 0, \quad \delta>\alpha(2/p-1). \tag{3}$$

Companion theorems to (2) and (3) are:

$$\int_{Q_k} S_*^\alpha(x,f)dx \leq A\int_{Q_k}|f(x)|(\log^+|f(x)|)^2dx+B, \tag{2'}$$

$$\lim_{R\to\infty}\int_{Q_k}|S_R^\alpha(x,f)-f(x)|dx = 0 \tag{3'}$$

if

$$\int_{Q_k}|f(x)|\log^+|f(x)|dx<\infty.$$

On strong summability, the author first proves that if $f\in L_p(Q_k)$, $1<p\leq 2$, then

$$\lim_{R\to\infty} R^{-1}\int_0^R|S_u^\delta(x,f)-f(x)|^2du = 0 \tag{4}$$

for almost every x, if $\delta>\alpha(2/p-1)-1/p'$, $1/p+1/p'=1$, a result not included in (3). Of particular interest is his result that if $f\in L_1(Q_k)$, then (4) holds almost everywhere for the critical exponent $\delta=\alpha$, which generalizes to several variables an important theorem of Marcinkiewicz [J. London Math. Soc. **14** (1939), 162–168; MR **1**, 11].

The novelty of the author's method consists in a skilful application of an interpolation theorem for an analytic family of operators due to himself [Trans. Amer. Math. Soc. **83** (1956), 482–492; MR **18**, 575]. The definition of S_R^δ is extended to complex δ so as to make it an analytic function in δ [such an extension in itself is not new; cf. S. Bochner and K. Chandrasekharan, Amer. J. Math. **71** (1949), 50–59; MR **10**, 369], and a suitable strip $a\leq \mathrm{Re}\,\delta\leq b$ is then considered. The line $\mathrm{Re}\,\delta=a$ is made to correspond to an L_2 result which can be deduced by using the Parseval relation, while the line $\mathrm{Re}\,\delta=b$ is made to correspond to an L_1 result which can be obtained by a direct estimate. The author's interpolation theorem then furnishes an argument of the Phragmén-Lindelöf type, which enables one to obtain L_p results on the intermediate lines $\mathrm{Re}\,\delta=c$, $a<c<b$. From the L_p results the author obtains results "near L_1" by delicate limiting arguments. *K. Chandrasekharan* (Bombay)

4332:

Rudin, Walter. Idempotent measures on Abelian groups. Pacific J. Math. **9** (1959), 195–209.

Let G be a locally compact abelian group with the Haar measure m_G and with the character group Γ. Let $\mathscr{I}(G)$ be the set of all idempotent Radon measures of G, i.e., $\mu\in\mathscr{I}(G)$ if $\mu*\mu=\mu$. The author gives various properties of μ in $\mathscr{I}(G)$ for locally compact or compact G. The first main result is a reduction to compact subgroups of locally compact G, which reads: for each $\mu\in\mathscr{I}(G)$, there exists a compact subgroup K of G on which μ is concentrated, i.e., $|\mu|(K)=\|\mu\|$ $(=|\mu|(G))$ where $|\mu|$ is the measure of total variation of μ. In the following, assume G to be compact. The author says in effect that $\mu\in\mathscr{I}(G)$ is irreducible if the smallest compact subgroup K of G on which μ is concentrated satisfies that $|\mu|(H)=0$ for every compact subgroup H of K with $m_K(H)=0$. The second main result is an irreducible decomposition: for any $\mu\in\mathscr{I}(G)$, there exist irreducible measures $\mu_i\in\mathscr{I}(G)$ and integers a_i such that $\mu=\sum_{i=1}^n a_i\mu_i$. The author defines the coset ring of Γ to be the smallest ring of sets which contains all cosets of all subgroups of Γ. Denote by $S(\mu)$ for $\mu\in\mathscr{I}(G)$ the set $\{\gamma\in\Gamma;\ \hat{\mu}(\gamma)=1\}$, where $\hat{\mu}$ is the Fourier-Stieltjes transform of μ. Then he remarks that every member of the coset ring of Γ is $S(\mu)$ for some $\mu\in\mathscr{I}(G)$. He states a conjecture for the converse case of this fact: for $\mu\in\mathscr{I}(G)$, $S(\mu)$ belongs to the coset ring of Γ. The last main result of this paper is the proof of this conjecture for the finite-dimensional torus groups [the one-dimensional case was previously proved by H. Helson, Proc. Amer. Math. Soc. **4** (1953), 686–691; **6** (1955), 235–242; MR **15**, 309; **17**, 597]. (The author states in the postscript that after the completion of this paper, P. J. Cohen has succeeded in proving the conjecture in its full generality [Bull. Amer. Math. Soc. **65** (1959), 120–122].) *H. Umegaki* (Tokyo)

INTEGRAL TRANSFORMS AND OPERATIONAL CALCULUS

See also 4312, 4665.

4333:

★Mikusiński, Jan. Operational calculus. International Series of Monographs on Pure and Applied Mathematics, Vol. 8. Pergamon Press, New York-London-Paris-Los Angeles; Państwowe Wydawnictwo Naukowe, Warsaw; 1959. 495 pp. $15.00.

The first (Polish) edition of Mikusiński's *Operational calculus* was published in 1953 [Polskie Towarzystwo Matematyczne, Warsaw; MR **16**, 243]. A Russian translation appeared in 1956 [Izdat. Inostr. Lit., Moscow; MR **18**, 575]; a second, revised and enlarged, Polish edition in 1957; and a German translation in the same year [VEB Deutscher Verlag Wiss., Berlin; MR **20** #1230]. The present, English, edition is based largely on the second Polish edition but it also contains over 100 pages of new material. Since the first Polish edition was reviewed in detail, it will be sufficient to enumerate the major changes which have been made since the appearance of that edition.

In Part I the material on discontinuous functions is considerably expanded, and there is a new chapter (written in collaboration with S. Drobot) on applications to the statics of beams.

The former Part II is replaced by two Parts. Part II is entitled "Sequences and series of operators" and is largely

new; it contains also a chapter on applications to difference equations. Part III contains the presentation of operational differential calculus.

Part IV (the former Part III) contains a new section on mixed problems in partial differential equations. Part V corresponds to the former Part IV.

Part VI is entitled "Appendix" and contains 112 pages of new material. Some of the chapters in this Part are simply appendices to appropriate chapters in the main body of the book. Other chapters, however, are on a far more advanced level than the main body of the book and demand considerable mathematical maturity, and some knowledge of abstract analysis, of the reader. These chapters establish the connection of Mikusiński's theory with the theory of distributions, discuss topological vector spaces and prove that Mikusiński's operators do not form a Banach space, define operators in terms of abstract algebra, etc. Among other topics taken up in this Part are: Laplace transforms, a class of Dirichlet series, power series of operators, some exponential functions, and a general theory of differential equations whose coefficients are operators.

Part VII corresponds to the former Part V, "Formulas and tables".

The translation was prepared, and the book was printed, in Poland. *A. Erdélyi* (Pasadena, Calif.)

4334:

Laha, R. G.; and Lukacs, Eugene. On distributions whose moments are majorated by the moments of a known distribution. J. Austral. Math. Soc. **1** (1959/61), part 1, 113–115.

Let F_1 and F_2 be two distribution functions with moments $\alpha_k^{(1)}$ and $\alpha_k^{(2)}$ and characteristic functions f_1 and f_2. If f_2 is analytic and $|\alpha_k^{(1)}| \leq |\alpha_k^{(2)}|$ for all k then f_1 is also analytic, and its circle of regularity about zero must be at least as large as that of f_2, although the strips of regularity of f_1 and f_2 may only overlap. If f_2 is an entire function then so is f_1, and if f_2 is of finite order then the order of f_1 is no larger.

R. P. Boas, Jr. (Evanston, Ill.)

4335:

Weston, J. D. Characterizations of Laplace transforms and perfect operators. Arch. Rational Mech. Anal. **3**, 348–354 (1959).

In an earlier paper the author introduced the class of "perfect operators" as a basis for Laplace transform calculus [Rend. Circ. Mat. Palermo (2) **6** (1957), 325–333; MR **21** #2871; compare Proc. Roy. Soc. London Ser. A **250** (1959), 460–471; MR **21** #2875]. He called a function p on $(0, \infty)$ perfect if it is infinitely differentiable, with $p^{(k)}(0+)=0$ and $p^{(k)}$ of at most exponential growth, $k=0, 1, 2, \cdots$. An operator A was called perfect if it takes every perfect function p into a perfect function Ap, with Laplace transform of the form $\bar{A}(s)\bar{p}(s)$ where $\bar{A}$ is holomorphic in a half-plane $\mathrm{Re}\, s > c$ and independent of p. In the present note the author characterizes the Laplace transforms of the perfect operators: they are the (holomorphic) functions $\bar{A}(s)$ which are bounded by a power of $|s|$ in some right half-plane. It follows that the perfect operators correspond exactly to the finite order distribution derivatives of the continuous functions of at most exponential growth on $[0, \infty)$. Thus Laplace transform theory based on perfect operators has exactly the same scope as Laplace transform calculus based on distributions. [Compare the reviewer's papers in Nederl. Akad. Wetensch. Proc. Ser. A **58** (1955), 368–389, 483–503, 663–674; MR **17**, 63, 354, 594].

J. Korevaar (Madison, Wis.)

4336:

Jain, Mahendra Kumar. Integral representation of Meijer transforms. Proc. Nat. Inst. Sci. India. Part A **23** (1957), 323–340.

The author calls

$$\phi(s) = s \int_0^\infty e^{-st/2}(st)^{-k-1/2} W_{k,m}(st) f(t)dt$$

the Meijer transform of $f(t)$. He uses several known integral representations or integral relations for $W_{k,m}$ to represent $\phi(s)$ as an iterated integral which is reduced to a single integral by interchanging the orders of integration. There are 10 general formulas obtained in this manner, with a number of accompanying examples. In a second group of 4 general formulas it is assumed that some function closely related to $\phi(s)$ is a Laplace transform of a function, and $f(t)$ is expressed as an integral involving that function.

A. Erdélyi (Pasadena, Calif.)

4337:

Karklin', I. V. Justification of certain formulas of the transformation of Êfros. Latvijas Valsts Univ. Zinātn. Raksti **8** (1956), no. 2, 81–91. (Russian. Latvian summary)

The expression $p \int_0^\infty e^{-pt}dt \int_0^\infty \psi(x, t)F(x)dx$ is called the Êfros transform of $F(x)$ with kernel $\psi(x, t)$, if ψ is a function such that $p \int_0^\infty e^{-pt}\psi(x, t)dt = e^{-xq(p)}q(p)u(p)$. This transform is formally equal to $u(p)f\{q(p)\}$ where $f(p) = p \int_0^\infty e^{-pt}F(t)dt$. It is proved in this paper that the formal result is true if $u(p)=p^{\nu+1/2}$, $q(p)=p^{1/2}$ when F satisfies certain conditions. It is also stated to be true when $u(p)=p^{\nu+1}/(p+bp^{1/2})$, $q(p)=p+bp^{1/2}$; $u(p)=p^2/(p^2+a^2)$, $q(p)=(p^2+a^2)^{1/2}$; $u(p)=p\{(p^2+a^2)^{1/2}-p\}^\nu/(p^2+a^2)^{3/2}$, $q(p)=(p^2+a^2)^{1/2}$ when $F(x)$ satisfies certain conditions corresponding to the form of the functions u and q.

F. Goodspeed (Vancouver, B.C.)

4338:

Mikolás, M. Differentiation and integration of complex order of functions represented by trigonometrical series and generalized zeta-functions. Acta Math. Acad. Sci. Hungar. **10** (1959), 77–124. (Russian summary, unbound insert)

The author starts from an integrable function f on $(0, 1)$ with Fourier series $f(u) \sim \sum \gamma_n e^{2n\pi iu}$ and defines the W_s-integral of f at x, where $\Re(s) > 1$, by

$$f_{(s)}(x) = \sum_{n=0}^{\infty} (2n\pi i)^{-s}(\gamma_n - \gamma_0)e^{2n\pi ix}.$$

Then for $\Re(s) > 1$ he obtains the integral representation

$$(*) \qquad f_{(s)}(x) = \int_0^1 f(x-t)[\mathfrak{Z}_s(t) - \mathfrak{Z}_s(x)]dt,$$

with kernel function

$$\mathfrak{Z}_s(u) = \sum_{n=1}^{\infty} 2(2n\pi)^{-s} \cos(2n\pi u - \tfrac{1}{2}\pi s)$$
$$= \Gamma(s)^{-1}\zeta(1-s, u),$$

where $\zeta(s, u)$ is the Hurwitz zeta function. He then defines $f_{[s_0]}(x)$ for other values s_0 as $\lim_{\sigma\to\sigma_0+} f_{[\sigma+it_0]}(x)$ provided that $f_{[s]}(x)$ can be continued analytically into an appropriate region. A large part of the paper is devoted to methods of performing the continuation under various hypotheses. For real negative s_0 the continuation can be made by Cesàro or Abel summability and yields the de La Vallée Poussin symmetric derivative of the appropriate order. The integral representation (*) can be used. If s is not a non-positive integer, $f_{[s]}(x)$ exists if and only if $\int_0^t f(x-t)t^{s-1}dt$ exists, and so the existence (not the value) of $f_{[s]}(x)$ depends only on the behavior of f in a left-hand neighborhood of x. Various sufficient conditions are given. A more detailed investigation for $0<\sigma\leq 1$ yields further formulas and further sufficient conditions; for example, if $f\in L^q$ $(q>1)$ then $f_{[s]}$ exists and is given by (*) for $\sigma>1/q$. Further criteria depend on analytic continuation of $f_{[s]}$ by means of power series. Riesz's generalization of Mittag-Leffler summability allows the determination of $f_{[s]}(x)$ in its Mittag-Leffler star. When f is analytic at x, then $f_{[s]}(x)$ exists for all s and is expressed as a contour integral. The semi-group properties $f_{[s][t]}=f_{[s+t]}$ and $f_{[s]}*g_{[t]}=(f*g)_{[s+t]}$ are discussed. Finally the author makes some remarks on $f_{[s]}$ as a generalized zeta function and on the possibility of extending his results to functions that are not Lebesgue integrable.

R. P. Boas, Jr. (Evanston, Ill.)

INTEGRAL AND INTEGRODIFFERENTIAL EQUATIONS

See also 4260, 4369, 4606, 4611, 4849.

4339:

Hildebrandt, T. H. On systems of linear differentio-Stieltjes-integral equations. Illinois J. Math. **3** (1959), 352–373.

Suppose n is a positive integer and BV the set to which g belongs only in case g is a function from the real numbers to a set of $n\times n$ matrices of bounded variation on every interval. Denote by I the identity function: $I(x)=x$ for every number x and by 1 the $n\times n$ identity matrix. Let ϕ denote the set of all F in BV with $F(0)=0$ and by H the set of all $n\times n$ harmonic matrices, i.e., all functions M from the ordered number pairs such that, for each number t, each of $M[I, t]$ and $M[t, I]$ is in BV and, if each of r and s is a number, $M(r, s)M(s, t)=M(r, t)$ and $M(t, t)=1$.

Generalizing a result of Wall in the case F continuous [Arch. Math. **5** (1954), 160–167; MR **15**, 801] (in the setting of a normed linear and complete space), MacNerney [Ann. of Math. (2) **61** (1955), 354–367; J. Elisha Mitchell Sci. Soc. **71** (1955), 185–200; MR **16**, 716; **18**, 54] showed that there exists a reversible transformation E from the set ϕ_1 of all F in ϕ for which $[\Delta^+F(u)]^2=[\Delta^-F(u)]^2=0$ onto the set H_1 of all M in H for which $\frac{1}{2}[M(t-, t)+M(t, t-)]=\frac{1}{2}[M(t, t+)+M(t+, t)]=1$, such that $M=E(F)$ only in case

$$M(s, t) = 1+\int_s^t dF\cdot M[I, t], \tag{1}$$

in the sense of mean Stieltjes integration; and only in case

$$M(s, t) = 1+\int_s^t M[s, I]\cdot dF, \tag{2}$$

in the same sense. He showed that the product integral ${}_s\prod^t(1+dF)$ exists if F is in ϕ and that $M=E(F)$ only in case $M(s, t)={}_s\prod^t(1+dF)$. Moreover, if each of f and g is in BV and $M=E(F)$, then the following are equivalent in the sense of mean Stieltjes integration:

$$f(t)-f(s)+\int_s^t dF\cdot f = g(t)-g(s), \tag{3a}$$

$$f(t) = M(t, s)f(s)+\int_s^t M[t, I]\cdot dg; \tag{3b}$$

as are also

$$f(t)-f(s)-\int_s^t f\cdot dF = g(t)-g(s), \tag{4a}$$

$$f(t) = f(s)M(s, t)+\int_s^t dg\cdot M[I, t]. \tag{4b}$$

The present author determines two reversible mappings E_1 and E_2 of subsets ϕ_1' and ϕ_2' of ϕ onto H by means of (1) and (2) respectively, as above, but with integration in a Lebesgue-Stieltjes sense; ϕ_1' is the set of all F in ϕ such that each of $1-\Delta^+F(u)$ and $1+\Delta^-F(u)$ is nonsingular and ϕ_2' is the set of all F in ϕ such that each of $1+\Delta^+F(u)$ and $1-\Delta^-F(u)$ is nonsingular. He characterizes E_1 and E_2 by means of product integrals which are limits of products of factors involving the matrices assumed nonsingular and their reciprocals. If $E_1(F)=E_2(G)$, then the continuous part of F is the continuous part of G and, if $M=E_1(F)$ and $N=E_2(F)$, then $M=N$ in case F is in ϕ_1'. If each of f and g is in BV, (3a) and (3b) are equivalent in the Lebesgue-Stieltjes sense in case $M=E_1(F)$ and (4a) and (4b) are equivalent in the same sense in case $M=E_2(F)$.

{For convenience in comparisons, the author's A has been replaced by $-F$ and his $B(v, u)$ by $M(u, v)$.}

H. S. Wall (Austin, Tex.)

4340:

Noble, B. The approximate solution of dual integral equations by variational methods. Proc. Edinburgh Math. Soc. **11** (1958/59), 115–126.

The author considers a class of dual integral equations of the type frequently encountered in the course of representing the solution of a mixed boundary value problem for a partial differential equation by means of superposition of separated solutions. In general, if one eliminates one of these equations, obtaining a single integral equation for the whole problem, one encounters divergent integrals, or is forced to modify the inversion problem. In this paper, no attempt is made to achieve such a reduction, but a variational principle is used so that the solution of each of the dual equations yields a stationary value for the variational problem. In the cases for which certain specific terms in the integral equations are of one sign, it is possible to obtain upper and lower bounds for functionals of the unknown solution. The theory is applied to the problem of finding the capacity of the parallel circular plate condenser.

G. E. Latta (Stanford, Calif.)

4341:

Germogenova, T. A. Some properties of solutions of integral equations on the half-line with kernel depending

on the difference of the arguments. Dokl. Akad. Nauk SSSR **126** (1959), 251–254. (Russian)

In the present work there are considered some properties of the solution of the non-homogeneous integral equation

$$f(x) = \int_0^\infty f(\xi)K(x-\xi)d\xi + g(x).$$

In particular, the author investigates the asymptotic behavior of the solution in so far as it depends on the function $g(x)$. He examines the nature of the solution near the boundary $x=0$, and finds approximations to the solution. By imposing certain stringent conditions on $g(x)$ and $K(x)$ the author is able to obtain, by the Wiener-Hopf method, an integral representation of the solution. This representation is used for the analysis of certain properties of the solution.

H. P. Thielman (Ames, Iowa)

4342:

Položiĭ, G. N. A method of solution of integral equations. Izv. Akad. Nauk SSSR. Ser. Mat. **23** (1959), 295–312. (Russian)

In the equation

$$(*) \qquad \int_a^b K(x, s)\varphi(s)ds = \mu\varphi(x) + F(x),$$

it is assumed that the functions K, F, φ are real and integrable and that

$$B^2 = \int_a^b \int_a^b K^2(x, s)dxds, \quad C^2 = \int_a^b K^2(x, s)ds,$$

$$D^2 = \int_a^b F^2(x)dx, \qquad E^2 = \int_a^b \varphi^2(x)dx$$

are finite. The characteristic values and functions of the symmetric kernel $K(x, s)$ are μ_i and $\varphi_i(x)$ and $\mu \neq \mu_i$. Then the author shows that as $m \to \infty$ the sequence of functions

$$\tilde{\varphi}_m(x) = -\frac{1}{\mu}F(x) + \sum_i c_i \frac{1-\nu_i^{m+1}}{1-\nu_i}\varphi_i(x), \quad |\nu_i| < 1,$$

converges uniformly to the solution $\varphi(x)$ of (*). Here,

$$c_i = \left(\frac{\mu_i^2}{\mu} - \mu_i\right)\frac{2}{\sigma}\int_a^b F(x)\varphi_i(x)dx, \quad \nu_i = 1 - \frac{2}{\sigma}(\mu_i - \mu)^2,$$

and σ is any number greater than $(B+|\mu|)^2$. The equation of the first kind ($\mu=0$ in (*)) is also considered.

J. F. Heyda (Cincinnati, Ohio)

4343:

Čerskiĭ, Yu. I. Solution of Riemann's boundary value problem in classes of generalized functions. Dokl. Akad. Nauk SSSR **125** (1959), 500–503. (Russian)

L'A. applique les résultats du travail, Mat. Sb. (N.S.) **41** (83) (1957), 277–296 [MR **19**, 559], pour trouver la solution du problème $f_+ = Af_- + g$ dans certains espaces de fonctionnelles linéaires, par exemple dans le dual de l'espace des fonctions hölderiennes définies sur une courbe simple, fermée, régulière dans le plan complexe.

G. Marinescu (Bucharest)

FUNCTIONAL ANALYSIS

See also 4175, 4209, 4223, 4231, 4294, 4333, 4391, 4392, 4394, 4401.

4344:

Fleischer, Isidore. Sur les modules localement polaires. C. R. Acad. Sci. Paris **248** (1959), 3094–3096.

"Les traits caractéristiques de la théorie des espaces vectoriels topologiques localement convexes sont formulés pour des modules topologiques. Le théorème de Mackey est démontré et les conditions pour la validité de l'analogue du théorème d'Hahn-Banach sont étudiées. Applications aux espaces vectoriels sur un corps topologique, en particulier, sur un corps valué." (Résumé de l'auteur)

M. Jerison (Lafayette, Ind.)

4345:

Pták, Vlastimil. Completeness and the open mapping theorem. Bull. Soc. Math. France **86** (1958), 41–74.

In the work of the past decade on locally convex topological linear spaces, an important development has been clarification of the notion of completeness and of its role in certain fundamental theorems. In the present expository paper, the author supplies a lucid analysis of some of his own contributions. The material discussed will not be described here, since most of it has already appeared in print (sometimes in less refined form), especially in the author's paper in Czechoslovak Math. J. **3** (78) (1953), 301–364 [MR **16**, 262]. However, his exposition can be strongly recommended to all readers who are interested in the subject, as he uses very effectively a method of presentation which can best be described in his own words as follows: "In this paper, we try to follow the ideas as they occurred to the author. This way of publishing results—so carefully avoided by many authors—is, we feel, the best for an article of this kind. The reader will find that, in many cases, a shorter proof might have been given. Nevertheless, we feel that even a longer proof is justified if it provides further insight into the matter." It may be remarked also that the present paper is in English while that cited above is in Russian with English summary.

V. L. Klee, Jr. (Copenhagen)

4346:

Deprit, André. Un type d'homomorphismes d'espaces localement convexes séparés. Ann. Soc. Sci. Bruxelles. Ser. I **72** (1958), 5–13.

Let E, F and G be locally convex topological linear spaces over the real or complex field. Let (E, F) be the set of all continuous linear mappings of E into F and $\tau(E, F)$ the subset consisting of all mappings u such that $u^{-1}(0)$ has a topological supplement in E, u is a homomorphism of E into F and $u(E)$ has a topological supplement in F. It is shown that if $u \in \tau(E, F)$, $v \in (E, F)$ and v is a compact operator then $u+v \in \tau(E, F)$. Let $v \in \tau(F, G)$ with $v^{-1}(0)$ finite-dimensional and let $u \in (E, F)$. Then $v \circ u \in \tau(E, G)$ if and only if $u \in \tau(E, F)$. If $w \in \tau(G, E)$ and its range has finite co-dimension, then $u \circ w \in \tau(G, F)$ if and only if $u \in \tau(E, F)$. Related properties of the adjoint operators to such mappings are investigated.

B. Yood (Eugene, Ore.)

4347:

Grünbaum, B. On a problem of S. Mazur. Bull. Res. Council Israel. Sect. F **7** (1957/58), 133–135.

Mazur's problem is: "Does there exist a 3-dimensional Banach space with the property that every 2-dimensional Banach space is isometric to a subspace of it?" This is equivalent to: "Does there exist a 3-dimensional convex body K with center O, such that any planar centrally symmetric convex body C is affinely equivalent to the intersection of K and a plane through O?" The author proves by simple means that the answer is negative. It is remarked that the general problem (replace "3" by "finite" and "2" by "n") has also been answered in the negative by C. Bessaga [Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 97–101].

Z. A. Melzak (Murray Hill, N.J.)

4348:

Gorin, E. A.; and Mityagin, B. S. Systems of norms in a countably-normed space. Uspehi Mat. Nauk **13** (1958), no. 5 (83), 179–184. (Russian)

The paper deals with complete countably-normed linear topological spaces. The linear topological space Φ with elements $\varphi, \cdots$ is countably-normed if its topology is given by a sequence of consistent non-decreasing norms $\|\varphi\|_1 \leq \|\varphi\|_2 \leq \cdots$. Φ is complete if $\Phi = \bigcap_p \Phi_p$, where Φ_p is the completion of Φ in the norm $\| \ \|_p$. The dual space Φ^* contains only elements f belonging to $\Phi_p{}^*$ for some p, and $\|f\|_p \geq \|f\|_{p+1} \geq \cdots$. Thus, $\|f\|_\infty = \lim_p \|f\|_p$ exists for each $f \in \Phi^*$. Solving a problem posed by Silov, the authors prove: If every closed bounded set in Φ is compact, then there exist systems of norms $\{\| \ \|_p\}$, $\{\| \ \|_p'\}$, giving the topology of Φ and such that $\|f\|_\infty = 0$ for each $f \in \Phi^*$, while $\|f\|_\infty' > 0$ for each $f \in \Phi^*$, $f \neq 0$. In connection with the proof of this theorem the authors introduce the space Φ_∞ of elements $\varphi \in \Phi$ for which the sequence $\|\varphi\|_1, \|\varphi\|_2, \cdots$ is bounded, with norm defined as $\|\varphi\|_\infty = \lim_p \|\varphi\|_p$. Φ_∞ is a Banach space, and if $\Phi = \Phi_\infty$ then Φ can be normed. In general, Φ_∞ does not characterize Φ: if Φ is separable then there exist norms $\{\| \ \|_p\}$, $\{\| \ \|_p'\}$, giving the topology of Φ and such that $\Phi_\infty\{\| \ \|_p\}$ is dense in Φ, while $\Phi_\infty\{\| \ \|_p'\}$ is trivial (consists of 0 only). If every closed bounded set in Φ is compact then Φ_∞ is trivial if and only if $\|f\|_\infty = 0$ for each $f \in \Phi^*$.

M. Golomb (Lafayette, Ind.)

4349:

Anderson, Frank W.; and Blair, Robert L. Characterizations of certain lattices of functions. Pacific J. Math. **9** (1959), 335–364.

Let K denote a conditionally complete, dense-in-itself chain with neither a first nor a last element; X a compact Hausdorff space containing at least two points. Let $C(X, K)$ be the lattice of all continuous functions on X to K, when K is endowed with its interval topology. A sublattice L of $C(X, K)$ is called a characterizing sublattice, if for each pair of distinct points x, y of X and each pair of functions f, g in L, there is an $h \in L$ such that $h(x) < f(x)$ and $h(y) > g(y)$. Among the preliminary results, it is proved that the topology of X is completely determined by the lattice structure of any one characterizing sublattice of $C(X, K)$. If Q is an order-dense subchain of K, then a sublattice L of $C(X, K)$ is called a Q-sublattice in case L contains all constant functions with values in Q. The main problem solved here is to find a necessary and sufficient set of conditions in order that a lattice be isomorphic to a characterizing Q-sublattice of $C(X, K)$ for some compact space X.

The authors' solution to this problem requires the following definitions. If L is a distributive lattice and if $\alpha, \beta \in L$, then we write $\alpha \dashv \beta$ in case $\alpha < \beta$ and, for any $f, g \in L$, the following two conditions hold: (i) if $f \vee g \geq \beta$ and $g \leq \alpha$, then $f \geq \beta$; (ii) if $f \wedge g \leq \alpha$ and $g \geq \beta$, then $f \leq \alpha$. A non-empty subset Q of a distributive lattice L is called a separating chain in L in case, relative to $\dashv$, Q is a chain having neither a first nor a last element. L is said to be a C_1-lattice relative to Q, if the following four conditions are satisfied: (a) Q is a separating chain in L; (b) if $\beta, \gamma \in Q$ with $\beta < \gamma$, then for any given $\alpha, \delta \in Q$ and $f \in L$, there exist $k, l, m \in L$ such that $k \wedge l \leq \alpha$, $f \vee l \geq \beta$, $f \wedge m \leq \gamma$, $k \vee m \geq \delta$; (c) if $f \in L$, $\alpha \in Q$ and if $f \leq \beta$ for all $\beta > \alpha$ in Q, then $f \leq \alpha$; (d) if $f, g \in L$ are such that for any $\alpha \in Q$ and $h \in L$, $h \wedge g \leq \alpha$ implies $h \wedge f \leq \alpha$, then $f \leq g$. If L is a C_1-lattice relative to a separating chain Q, then L is said to be bounded, if for each $f \in L$, there exist $\alpha, \beta \in Q$ such that $\alpha \leq f \leq \beta$. If Q is a chain, we shall denote by $\bar{Q}$ the completion of Q by cuts, and by $\hat{Q}$ the conditionally complete subchain of $\bar{Q}$ obtained by removing those extreme points of $\bar{Q}$, if any, which are not in Q. With these definitions, the authors proved the following theorem. Let Q be an order-dense subchain of K. If a lattice L is isomorphic to a characterizing Q-sublattice of $C(X, K)$ for some compact space X, then the inverse image of Q in L is a separating chain in L relative to which L is a bounded C_1-lattice. Conversely, if L is a bounded C_1-lattice relative to Q, then there exists an isomorphism of L onto a characterizing Q-sublattice of $C(X, \hat{Q})$ for some compact space X. From this result, two characterizations of the entire lattice $C(X, K)$ are obtained by imposing, in two different ways, a completeness condition.

For the case when K is the chain R of real numbers, the authors point out the difference between their results and the previous characterization of I. Kaplansky [Amer. J. Math. **70** (1948), 626–634; MR **10**, 127] for certain sublattices of $C(X, R)$, and those of L. J. Heider [Duke Math. J. **23** (1956), 297–301; MR **18**, 52], A. G. Pinsker [Uspehi Mat. Nauk (N.S.) **12** (1957), 226–229; MR **19**, 7] for the entire lattice $C(X, R)$. It is also indicated how the results can be used to deduce the characterization of $C(X, R)$ given by M. H. Stone [Proc. Nat. Acad. Sci. U.S.A. **27** (1941), 83–87; MR **2**, 318] as a lattice-ordered group, and that by the reviewer [Ann. of Math. (2) **51** (1950), 409–427; MR **11**, 525] as a partially ordered additive group.

Ky Fan (Notre Dame, Ind.)

4350:

Buck, R. Creighton. Bounded continuous functions on a locally compact space. Michigan Math. J. **5** (1958), 95–104.

Let $C^*[X : E]$ denote the set of bounded continuous functions on a locally compact space X to a complete locally convex real linear space E; when E is the real line, this is shortened to $C^*[X]$. $C^*[X : E]$ is a $C^*[X]$ module in a natural way. The "strict" topology β on $C^*[X : E]$ may be defined as the locally convex topology obtained from the semi-norms

$$\|f\|_{\phi,\nu} = \|\phi f\|_\nu = \sup_{x \in X} |\phi(x) f(x)|,$$

where ν ranges over an index set for the semi-norms describing the topology on E, and ϕ ranges over the set of functions in $C^*[X]$ that vanish at infinity. β lies between the compact-open and uniform topologies.

The author gives several properties of the space $C^*[X:E]$ with the topology β, e.g., that it is topologically complete, generalizing his earlier results [Proc. Amer. Math. Soc. **3** (1952), 681–687; MR **14**, 290]. It is shown that the strict dual space of $C^*[X]$ is $\mathfrak{M}[X]$, the space of bounded Radon measures on X. This is similar to the known result that for compact X the uniform dual of $C^*[X]$ is $\mathfrak{M}[X]$—which, however, does not carry over to non-compact X. Two Stone-Weierstrass type theorems are also obtained: A. If $\mathfrak{A}$ is a β-closed subalgebra of $C^*[X]$ that separates points of X and contains a function vanishing nowhere, then $\mathfrak{A}=C^*[X]$. B. Let X be metrizable and E finite-dimensional. If $\mathfrak{A}$ is a β-closed submodule of $C^*[X:E]$ and $\mathfrak{A}(x)=E$ for each $x\in X$, then $\mathfrak{A}=C^*[X:E]$. In B, it is not assumed that $\mathfrak{A}$ is an algebra.

{The remarks following Lemma 3 are inaccurate; however, this has no effect on the rest of the paper. The author notes that on p. 99, l. 2* and p. 100, l. 7, "$C_\infty[X]$" should read "$C^*[X]$"; and the last displayed formula on p. 102 should read "$f_N(x)=\sum_{j\in N}\psi_j(x)g_j(x)$ (N a finite set)."}

C. W. Kohls (Urbana, Ill.)

4351:

Geba, K.; and Semadeni, Z. On the embedding of one space of continuous functions into another. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 187–189. (Russian summary, unbound insert)

For two compact Hausdorff spaces Ω and Ω_0, let $C(\Omega)$ and $C(\Omega_0)$ denote the Banach lattices of all real-valued continuous functions on Ω and Ω_0 respectively. Denote by e, e_0 the constant functions 1 on Ω and Ω_0 respectively. Assume that T is a linear one-to-one, norm-preserving and isotone transformation from $C(\Omega_0)$ onto a subspace X_0 of $C(\Omega)$. Let Ω_1 denote the set of all points $t\in\Omega$ satisfying $(Te_0)(t)=e(t)$ and $T(x\vee y)(t)=(Tx\vee Ty)(t)$ for all $x, y\in C(\Omega_0)$. The following results are announced without proof, (I) $\|x\|=\sup\{|(Tx)(t)|: t\in\Omega_1\}$ for every $x\in C(\Omega_0)$. (II) There exists a continuous mapping from Ω_1 onto Ω_0. (III) If there exists a non-negative projection P of norm 1 from $C(\Omega)$ onto X_0, then Ω_0 is homeomorphic to a closed subset Ω_P of Ω, and P corresponds to the operation of restricting the functions to Ω_P. (IV) If $Te_0=e$ and $T(x\vee y)=Tx\vee Ty$ for all $x, y\in C(\Omega_0)$, and if there exists a non-negative projection from $C(\Omega)$ onto X_0, then Ω_0 is homeomorphic to a retract of Ω.

Ky Fan (Notre Dame, Ind.)

4352:

Shirai, Tameharu. Sur les topologies des espaces de L. Schwartz. Proc. Japan Acad. **35** (1959), 31–36.

L'A. considère une pseudotopologie définie par une opération $A\to\bar{A}$, dont il ne précise que la propriété de monotonie et définit une nouvelle opération $A\to\tilde{A}=\bigcap_{F\supset\mathcal{F}\supset A}F$. Il démontre que dans le cas de l'espace $\mathcal{D}$ de L. Schwartz cette opération définit une topologie strictement plus fine que la topologie de L. Schwartz.

G. Marinescu (Bucharest)

4353:

Urbanik, K. A theorem on distributions integrable with even power. Studia Math. **17** (1958), 323–333.

For a distribution T of one variable, $|T|^{2p}$ will denote the distribution of $2p$ variables

$$T\times\cdots\times T\times\overline{T}\times\cdots\times\overline{T}\quad(T, p\text{ times};\ \overline{T}, p\text{ times}),$$

where $(\overline{T},\varphi)=\overline{(T,\bar{\varphi})}$. Also $\int_a^b S$ is defined by $(\int_a^b S,\varphi)=(S,\varphi_{a,b})$, where

$$\varphi_{a,b}(x_1,\cdots,x_n)=\int_a^b\varphi(x_1-t,\cdots,x_n-t)dt.$$

Finally, $\int_{-\infty}^{\infty}S=\lim_{a,b\to\infty}\int_{-a}^{b}S$, in the sense of distributions, if the limit exists.

Theorem: $T\in\mathcal{D}_{L^{2p}}\Leftrightarrow\int_{-\infty}^{\infty}|T|^{2p}$ exists.

J. Feldman (Berkeley, Calif.)

4354:

Sebastião e Silva, J. Les fonctions analytiques comme ultra-distributions dans le calcul opérationnel. Math. Ann. **136** (1958), 58–96.

This paper is a continuation of the work of L. Schwartz in defining the Fourier transform of very general kinds of distributions. Schwartz gave a method of defining the Fourier transform for tempered distributions; as such the transform remained within the realm of real analysis. The reviewer [Ann. of Math. (2) **63** (1956), 129–159; MR **17**, 876] showed that by going into the complex domain one could define the Fourier transform of an arbitrary distribution of Schwartz. The Fourier transform of a distribution may no longer be a distribution of Schwartz; it is, in general, an element of the dual of a space of complex analytic functions, which we may call an ultra-distribution.

In the present paper the author defines the Fourier transform on certain other spaces of distribution or ultra-distributions. He then shows that most of the usual properties of Fourier transform are preserved, and he gives representations for his ultra-distributions. Finally the author produces a space of ultra-distributions which is invariant under Fourier transform.

L. Ehrenpreis (New York, N.Y.)

4355:

Ehrenpreis, Leon. Analytic functions and the Fourier transform of distributions. II. Trans. Amer. Math. Soc. **89** (1958), 450–483.

Cet article est la continuation du travail de l'A. sur la transformation de Fourier définie pour des distributions quelconques [Ann. of Math. (2) **63** (1956), 129–159; MR **17**, 876]. Maintenant l'A. étend sa définition au cas de distributions à valeurs dans le dual d'un espace l.c. complet E, de la façon suivante.

Soit $\mathcal{D}(E)$ l'espace des fonctions φ à valeurs dans E, indéfiniment dérivables sur R^n et à support compact. On peut définir, suivant la formule classique, la transformation de Fourier, $\mathcal{F}$, pour les fonctions $\varphi\in\mathcal{D}(E)$, et on démontre, par une généralisation du théorème de Paley-Wiener, que les images $\mathcal{F}(\varphi)$ de ces fonctions s'identifient aux fonctions entières à valeurs dans E, à "décroissance rapide" sur R^n et du "type exponentiel" par rapport aux parties imaginaires. L'espace des fonctions $\mathcal{F}(\varphi)$, avec $\varphi\in\mathcal{D}(E)$, est désigné par $\mathbf{D}(E)$ et muni d'une topologie qui rend $\mathcal{F}$ un isomorphisme de $\mathcal{D}(E)$ sur $\mathbf{D}(E)$. Alors on définit aussitôt, par dualité, un isomorphisme $\mathbf{F}$ de $\mathcal{D}(E)'$ sur $\mathbf{D}(E)'$: $\mathbf{F}$ sera la transposée de $\mathcal{F}^{-1}$ et on peut la considérer comme la transformation de Fourier étendue aux distributions sur R^n à valeurs dans E'. {Ici le rapporteur trouve quelques difficultés d'interprétation. Le théorème 10 de cet article dit que le dual fort de l'espace $\mathcal{D}(E)$ est isomorphe à l'espace $\mathcal{L}_b(\mathcal{D};E)\simeq\mathcal{D}'\hat{\otimes}E'$, des distributions sur R^n à valeurs dans E', et l'A. présente ce résultat

comme une variante du "théorème des noyaux" de L. Schwartz. Or cela n'est pas évident pour le rapporteur. En outre, quoique l'A. soutienne l'équivalence des deux méthodes, il semblerait plus naturel et facile de définir directement les espaces $\mathscr{D}'(E) \simeq \mathscr{L}_b(\mathscr{D}; E)$, $\mathbf{D}'(E) \simeq \mathscr{L}_b(\mathbf{D}; E)$ et de poser par définition $(\mathbf{F}T)(\varphi) = T(\mathscr{F}^{-1}\varphi)$, pour $T \in \mathscr{D}'(E)$ et $\varphi \in \mathbf{D}$, suivant l'idée même de L. Schwartz pour les distributions tempérées; cela définirait $\mathbf{F}$ pours toutes les distributions à valeurs dans E arbitraire.}

Dans le dernier paragraphe l'A. applique ses résultats à l'étude de fonctions analytiques dans une bande $a < \mathscr{I}(z) < b$ et ayant des types assez généraux de distribution limite sur la frontière (e.g. des distributions $\in L^p$, etc.). Il démontre que ces fonctions sont univoquement déterminées par leurs distributions de frontière, et généralise, aux cas considérés, le théorème de Phragmén-Lindelöf, d'après lequel la majoration $|f(z)| \leq M$ sur les droites $\mathscr{I}(z) = a$, $\mathscr{I}(z) = b$ implique, sous certaines conditions à l'infini, la même relation sur la bande $a < \mathscr{I}(z) < b$.

J. Sebastião e Silva (Lisbon)

4356:

Slowikowski, W. A theory of extensions of map-systems. I. Fund. Math. **46** (1959), 243–275.

Dans cet article l'A. réproduit, sous une forme un peu différente, le "théorème général d'extension algébrique" établi par le rapporteur dans sa construction axiomatique de la théorie des distributions [Univ. Lisboa Revista Fac. Ci. A (2) **4** (1954–55), 79–186; **5** (1956), 169–170; MR **17**, 766; voir Notes Finales, I, pp. 177–179]. Il semble toutefois que l'A. n'ait peut-être pas eu la possibilité de lire ce travail, qu'il cite seulement dans la Bibliographie de son article.

D'après l'A., un système opératoire ("operator-system") est tout couple (S, X) où: (a) X est un groupe commutatif à notation additive; (b) S est un semigroupe commutatif à notation multiplicative, muni d'élément unité, I; (c) à tout $A \in S$ correspond un sous-groupe X_A de X, et à tout couple $A \in S$, $x \in X$ correspond un élément de X noté Ax, de telle façon que: (1) l'application $x \to Ax$ est un homomorphisme de X_A sur X; (2) quels que soient A, $B \in S$, on a $X_{AB} \subset X_B$, $B(X_{AB}) \subset X_A$ et $(AB)x = A(Bx)$ pour tout $x \in X_{AB}$. Un système opératoire est dit algébriquement fermé, si l'on a $X_A = X$ pour tout $A \in S$. Si S est un sous-groupe de S', on appelle homomorphisme simple d'un système $\mathfrak{A} = (S, X)$ dans un autre $\mathfrak{A}' = (S', X')$, tout homomorphisme h de X dans X' tel que: $h(X_A) \subset X_A'$ et $h(Ax) = A(hx)$, pour tout $A \in S$; si en outre l'application h de X dans X' est biunivoque, on dit que h est un isomorphisme de $\mathfrak{A}$ dans $\mathfrak{A}'$. On appelle idéal du système $\mathfrak{A} = (S, X)$ tout opérateur $\mathfrak{I}$ faisant correspondre à chaque $A \in S$ un sous-groupe $\mathfrak{I}(A)$ de X, de telle façon que:

$$\mathfrak{I}(B) \subset \mathfrak{I}(AB) \quad \text{et} \quad X_A \cap \mathfrak{I}(AB) = A^{-1}(\mathfrak{I}(B)).$$

Cela posé, la première partie du "théorème fondamental algébrique" peut s'énoncer comme suit: pour tout système $\mathfrak{A} = (S, X)$ et tout idéal $\mathfrak{I}$ de $\mathfrak{A}$, il existe un (et seulement un, à un isomorphisme près) système algébriquement fermé $\mathfrak{A}' = (S', X')$, et un homomorphisme simple h de $\mathfrak{A}$ dans $\mathfrak{A}'$, tels que: (i) $\mathfrak{I} = h^{-1}(\mathfrak{I}')$, où $\mathfrak{I}'$ est l'idéal minimal de A'; (ii) chaque élément y' de X' est de la forme $y' = Ax'$, avec $A \in S$, $x' = h(x)$, $x \in X$. Alors, h sera un isomorphisme, si et seulement si on a $\mathfrak{I}(I) = \{0\}$; dans ce cas, $\mathfrak{A}'$ peut être considéré comme une "extension" de $\mathfrak{A}$, si l'on identifie X à $h(X) \subset X'$. On peut reconnaître que cette partie du théorème équivaut essentiellement au théorème d'extension algébrique du rapporteur [loc. cit., pp. 98–99 et, surtout, pp. 177–179]. La deuxième partie correspond au "théorème d'homomorphisme" du rapporteur [loc. cit., pp. 114–115]. Comme application principale de ce théorème, l'A. indique le cas où X est l'espace des fonctions continues dans un domaine Ω de R^n et S est le semigroupe des opérateurs $D_1^{p_1} D_2^{p_2} \cdots D_n^{p_n}$ de dérivation; alors l'extension X' obtenue est l'espace des distributions (d'ordre fini) sur Ω. {Dans quelques notes à paraître dans Rend. Accad. Lincei, sur le calcul opérationnel d'opérateurs différentiels à coefficients variables, le rapporteur emploie les mêmes méthodes pour introduire d'autres fonctions généralisées, qu'il nomme "para-distributions".}

Dans l'Introduction l'A. annonce la généralisation (qui sera démontrée dans la partie II de cet article) de son théorème fondamental, au cas où S n'est plus commutatif et où ses éléments A sont des homomorphismes de sous-groupes de X dans X, mais non nécessairement sur X. Or cette généralisation surpasse essentiellement les résultats du rapporteur et aura probablement beaucoup d'intérêt.

J. Sebastião e Silva (Lisbon)

4357:

Tillmann, Heinz Günther. Zur Definition der analytischen Funktionale. Arch. Math. **10** (1959), 8–11.

Cet article est suscité par un résultat de D. Pisanelli [Bol. Soc. Mat. São Paulo **9** (1954), 1–71; MR **20** #1909, #1910]. D'après L. Fantappiè, une fonctionnelle analytique est toute application F d'un ouvert $\mathscr{A}$ de son espace $\mathscr{S}$ dans le corps complexe $\mathbf{C}$, telle que: (1) si f_λ est une "ligne analytique" traversant $\mathscr{A}$, alors $F[f_\lambda]$ est une fonction holomorphe de la variable complexe λ; (2) pour tout prolongement $\bar{f}$ de f, avec $f, \bar{f} \in \mathscr{A}$, on a $F[\bar{f}] = F[f]$. Or Pisanelli a montré que, si l'on remplace dans (1) la notion de "ligne analytique" par celle, plus générale, de "ligne quasi-analytique", la condition (2) est impliquée par (1). D'autre part, l'A. a remplacé dans $\mathscr{S}$ la topologie de Fantappiè par une topologie naturelle plus fine, métrisable, ne donnant pas lieu aux faits pathologiques signalés par le rapporteur [Tillmann, Abh. Math. Sem. Univ. Hamburg **21** (1957), 139–193; MR **20** #1911]; il a introduit alors une notion de "ligne analytique" basée sur la nouvelle topologie et conduisant à une définition de "fonctionnelle analytique", formellement analogue et pratiquement équivalente à celle de Fantappiè. Maintenant l'A. montre que, dans sa systématisation, la condition (2) est encore impliquée par (1), sous certaines restrictions imposées aux domaines $\mathscr{A}$.

J. Sebastião e Silva (Lisbon)

4358:

Salehov, D. V. A characteristic property of L_p spaces in the class of Orlicz spaces. Voronež. Gos. Univ. Trudy Sem. Funkcional. Anal. **1957**, no. 5, 39–51. (Russian)

Detailed proofs of results announced by the author previously [Dokl. Akad. Nauk SSSR **111** (1956), 948–950; MR **18**, 811].

G. G. Lorentz (Syracuse, N.Y.)

4359:

Lax, Peter D. Translation invariant spaces. Acta Math. **101** (1959), 163–178.

Let $(L_2^+)^q$ be the set of all q-dimensional (column-) vector-valued functions $\mathbf{f}=(f_1, \cdots, f_q)$ on $(-\infty, \infty)$ with complex-valued components $f_k \in L_2^+$, i.e. with f_k which are one-sided Fourier transforms:

$$f_k(x) = \int_0^\infty e^{ixt} g_k(t)dt, \quad g_k \in L_2(0, \infty).$$

A subspace R of $(L_2^+)^q$ is called "right translation invariant" (R.T.I.), if $\mathbf{f} \in R$ implies that $\mathbf{g}$, where $\mathbf{g}(x) = e^{ixa}\mathbf{f}(x)$, is also in R for all $t>0$. In essence the following important theorem is proved. Thm. A: To every non-trivial R.T.I. (closed) subspace R of $(L_2^+)^q$ corresponds a unique integer p, $1 \leq p \leq q$, and a $q \times p$ matrix-valued function F such that (a) $F \in H_\infty$, i.e. the entries of F are holomorphic and bounded in the upper half of the complex plane; (b) $F(t)$ is an isometry a.e., $-\infty < t < \infty$; (c) $R = F \cdot (L_2^+)^p$; i.e. for each $\mathbf{g} \in R$ we have $\mathbf{g}(t) = F(t)\mathbf{f}(t)$, a.e., where $\mathbf{f} \in (L_2^+)^p$. F is unique up to a constant $p \times p$ unitary matrix post-factor.

With $q=1$ and with the upper half plane replaced by the unit disk, this theorem reduces to Thm. IV of Beurling [Acta Math. **81** (1949), 239–255; MR **10**, 381]. Thm. A thus extends Beurling's in two ways: (1) from the discrete to the continuous parameter case; (2) from complex-valued to vector-valued functions.

The transition (1) can be made either by Cayley transformation or by a direct attack not appealing to the discrete theory. The author adopts the second alternative, following certain ideas of Beurling [op. cit.] and Karhunen [Ark. Mat. **1** (1950), 141–160; MR **11**, 607]. But unlike Beurling, who had to take recourse to Thm. B (below) and prove a chain of ancillary results to establish his Thm. IV, the author proves Thm. A directly by more efficient utilization of the unitary structure of the space L_2^+ and its orthogonal complement L_2^- (§ 2). {The reviewer is unable to see, however, why the function b_λ (written b) should be differentiable (p. 167). He also feels that the use made of the duals of the spaces L_2^+, L_2^- is dispensable.}

In § 3 results on the divisibility of complex-valued functions satisfying conditions (1), (2) of Thm. A are deduced. Thus the continuous parameter analogues of some of Beurling's lemmas emerge as corollaries. Also obtained is a new proof of the convolution theorem of Titchmarsh.

The proof of Thm. A for $q=1$ is shown (§ 4) to carry over for $q>1$ with suitable modifications to cover the plurality of dimension and possible degeneracies of rank. The techniques developed to handle the case $q=1$ thus serve in effecting the transition (2) as well as the transition (1).

At the end of § 2 the author deduces:

Thm. B. Every function in L_2^+ is the product of an inner function and an outer function.

However, an outer function ϕ is defined by the author as one for which ϕH_∞ is everywhere dense in L_2^+. This definition is equivalent to the usual one:

$$\log|\phi(i)| = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\log|\phi(t)|}{1+t^2}\,dt.$$

But this equivalence is not obvious: in the discrete case it depends on Beurling's Thm. I. Hence, to derive Thm. B, with the usual interpretation of the term "outer", from Thm. A one would have to prove a result akin to Beurling's Thm. I. This would make the proof rather long.

At the end of § 4 it is asserted that Thm. B extends to matrix-valued functions with entries in L_2^+, but neither enunciation nor proof is given. On the basis of a communication from the author, the reviewer may say that such an extension does hold, provided that rectangular-matrix factors are admitted and the term "outer" is interpreted in analogy with the author's definition for $q=1$. (The resulting factorization seems to be a different version of one recently obtained by the reviewer [C. R. Acad. Sci. Paris **249** (1959), 873–875] in which the factors are square matrices but a projection matrix of rank $\leq q$ occurs.)

The determination of the exact relationship between Thms. A and B, and between these and certain results in the theory of weakly stationary stochastic processes is still open. As of now it would seem that not much is gained by treating one theorem as a corollary of the other, and that direct attacks on each are more revealing.

P. Masani (Providence, R.I.)

4360:

Phillips, R. S. On a theorem due to Sz.-Nagy. Pacific J. Math. **9** (1959), 169–173.

Let $\{T_t\}$ be a strongly continuous semi-group of contraction operators in a Hilbert space H. Let L be the generator of $\{T_t\}$ and $J=(I+L)(I-L)^{-1}$. Let us call the contraction J the cogenerator of $\{T_t\}$. The author proves that the unitary B. Sz.-Nagy dilation $\mathbf{J}$ of J is the cogenerator of the unitary B. Sz.-Nagy dilation $\{\mathbf{T}_t\}$ of $\{T_t\}$. {Reviewer's remark: The author seems to be unaware of the paper by B. Sz.-Nagy and the reviewer [Acta Sci. Math. Szeged **19** (1958), 26–45; MR **21** #2188], in which this fact is already established.} *C. Foiaș* (Bucharest)

4361:

Kurepa, Svetozar. Semigroups of normal transformations in Hilbert space. Glasnik Mat.-Fiz. Astr. Društvo Mat. Fiz. Hrvatske. Ser. II **13** (1958), 257–266. (Serbo-Croatian summary)

Let G_+ be the additive semi-group of rational numbers of the form $r=j/2^k, j, k=0, 1, 2, \cdots$. Suppose $[N(r); r \in G_+]$ is a semi-group of normal operators on a Hilbert space H such that (i) 0 is not in the point spectrum of any $N(r)$, and (ii) $D=\bigcap D(r)$ is dense in H; here $D(r)$ denotes the domain of $N(r)$. In this case it is shown that $N(r)$ has a unique representation

$$N(r) = U(r)\exp(rA) = \exp(rA)U(r),$$

where A is self-adjoint and $[U(r); r \in G_+]$ is a semi-group of unitary operators. An analogous result is proved for semi-groups of normal operators $[N(t); \text{all } t \geq 0]$ such that (i) 0 is not in the point spectrum of any $N(t)$, (ii) $D=\bigcap D(t)$ is dense in H, and (iii) $\sup[\|N(t)f\|; t \in T] < \infty$ for each $f \in D$, where $T \subset [0, \infty)$ is such that $m_1(T+T)>0$. Since no continuity assumptions are made, nothing can be said of the continuity of $[U(t)]$. The paper also contains two results on semi-groups of bounded normal operators.

R. S. Phillips (Los Angeles, Calif.)

4362:

Mibu, Yoshimichi. On the spectral-resolutions of quasi-compact elements in a B^*-algebra. Proc. Japan Acad. **35** (1959), 209–212.

This paper deals with the theorem on the resolution of an element a of a Banach algebra A in terms of the connected components of its spectrum. The author proves that the theorem is valid if A is a commutative B^*-algebra

and 0 is the only limit point of the spectrum of a. This however is immediately evident from the functional representation of commutative B^*-algebras (which are C^*-algebras). *J. M. G. Fell* (Seattle, Wash.)

4363:

Suzuki, Noboru. Crossed products of rings of operators. Tôhoku Math. J. (2) **11** (1959), 113–124.

Let M be a finite factor with the invariant $C=1$ over a Hilbert space H, and G a group of automorphisms (i.e., *-automorphisms) of M. Then there is a faithful unitary representation u of G on H such that $a^\sigma = u_\sigma^* a u_\sigma$ for every $a \in M$ and $\sigma \in G$. Let $\mathfrak{H} = H \otimes l_2(G)$ be the Hilbert space of l_2-sequences of elements of H indexed by G. Then M induces a ring of operators $M \otimes I$ on $\mathfrak{H}$ and u induces a faithful unitary representation U of G on $\mathfrak{H}$ so that $U_\sigma \varphi_\tau = \psi_{\sigma\tau}$ and $U_\sigma^* A U_\sigma = A^\sigma$, where $\psi_{\sigma\tau} = u_\sigma \varphi_\tau$, $A = a \otimes I$, and $A^\sigma = a^\sigma \otimes I$. The W^*-algebra generated by the finite sums of the form $\sum_{\sigma \in G} A_\sigma U_\sigma$ is called the crossed product of M by the group G of automorphisms. It is shown that the various crossed products of M by G (arising from different u) are spatially isomorphic and the crossed product is a finite W^*-algebra with $C=1$. If M is a II_1-factor and G a group of outer automorphisms, then the crossed product is also a II_1-factor. That the crossed product may not be a factor is shown by an example. Finally, the author gives a sufficient condition that a W^*-algebra with $C=1$ be a crossed product of an appropriate subfactor and a group of automorphisms of the subfactor. *Ti Yen* (Chicago, Ill.)

4364:

★**Waelbroeck, Lucien. Structure des algèbres d'éléments réguliers.** Colloque d'algèbre supérieure, tenu à Bruxelles du 19 au 22 décembre 1956, pp. 251–259. Centre Belge de Recherches Mathématiques. Établissements Ceuterick, Louvain; Librairie Gauthier-Villars, Paris; 1957. 293 pp. 250 francs belges.

Let A be a commutative topological algebra with identity 1. The topological structure is assumed to be locally convex, T_2, quasi-complete, and the product ab separately continuous in a and b. An element a of A is "regular" in case $(a-s)^{-1}$ is defined and bounded for all complex s in some neighborhood of ∞. A^* is the set of regular elements of A. The spectrum of a regular element a is the nonempty compact set of singularities of the holomorphic function $(a-s)^{-1}$. In general, if $\{a_i \mid i \in I\}$ is a family of regular elements, the spectrum of $\{a_i \mid i \in I\}$ is the set of all complex-valued functions s_i such that for any finite subset $J \subset I$, and any polynomial P in $n(J)$ (= number of elements in J) variables, $P(s_{i_1}, \cdots, s_{i_{n(J)}})$ is in the spectrum of $P(a_{i_1}, \cdots, a_{i_{n(J)}})$. These functions s_i are topologized via simple convergence.

If $\mathfrak{F}$ is the family of all complex-valued functions z_i on I, a function f defined on $\mathfrak{F}$ is one that depends effectively on only finitely many $z_i: z_{i_1}, \cdots, z_{i_m}$. If $a = \{a_i \mid i \in I\}$ and if f is in the direct limit of the algebras of functions holomorphic on neighborhoods of the projection of the spectrum of a onto the coordinates $i_1, \cdots, i_m$, then $f(a)$ is definable in a natural manner so that the representation $f(z) \to f(a)$ takes z_i onto a_i and the constant 1 into the identity of A^*.

If $a = \{a_i \mid i \in I\}$ is a family of analytic generators of A^* then the representation: $f(z) \to f(a)$ is a representation of $\mathscr{H}$(spectrum a) (=the direct limit of the functions holomorphic in neighborhoods of spectrum a) onto A^*. Its kernel b is a subset of c, the set of functions vanishing on spectrum a.

If $b \in A^*$, choose $f \in \mathscr{H}$(spectrum a) so that $f(a) = b$, and define $\hat{b}(s)$ to be $f(s)$, thus providing a representation of A^* into the continuous functions on spectrum a. If $s = \{s_i\} \in$ spectrum a, then $\hat{a}_i(s) = s_i$, i.e., if $s = \{s_i\} \in$ spectrum a there is a representation σ of A^* onto the complexes such that $\sigma(a_i) = s_i$. Conversely if such a σ exists then $s \in$ spectrum a.

There is a 1-1 correspondence between spectrum a and the maximal ideals of A^*, and this provides an alternative view of the structure space of A^*.

B. R. Gelbaum (Minneapolis, Minn.)

4365:

Civin, Paul; and Yood, Bertram. Involutions on Banach algebras. Pacific J. Math. **9** (1959), 415–436.

Let B be a complex Banach algebra. By an involution on B is meant a conjugate linear anti-automorphism of period two. The authors conduct an extensive investigation of involutions on B. Examples are furnished where B is semi-simple, commutative, and has a unique involution, where B is semi-simple, commutative, and has no involution, and where B is semi-simple, not commutative, and has no involution.

An involution $x \to x^*$ is called (1) symmetric if every xx^* has a non-negative real spectrum, (2) Hermitian-real if every self-adjoint x has a real spectrum, (3) regular if no non-zero self-adjoint x has spectral radius 0, and (4) proper if no non-zero x has $xx^* = 0$. Typical of the results obtained connecting these conditions are the following. For a symmetric involution on a semi-simple B, continuity is equivalent to (3) and implies (4). If B is not commutative and has a continuous involution *, then B has non-denumerably many distinct continuous involutions which may be chosen to be symmetric (Hermitian-real) if * is. If B is semi-simple, then no symmetric involution commutes with a different continuous symmetric involution. If B is not commutative and is either a B^*-algebra or an H^*-algebra, then B has a symmetric involution with arbitrarily large norm (when considered as a linear operator on B). If B is a semi-simple annihilator algebra, then conditions (1), (2), and (4) are equivalent.

Let B be a commutative complex Banach algebra and let $x \to x'$ be a real linear anti-automorphism of period two. The authors present a number of results connecting the algebraic structure induced on B by $'$ and properties of the space of maximal regular ideals of B.

J. A. Schatz (Albuquerque, N.M.)

4366:

Bahtin, I. A. Non-linear equations with concave and uniformly concave operators. Dokl. Akad. Nauk SSSR **126** (1959), 9–12. (Russian)

The author continues his search [see Bahtin, same Dokl. **117** (1957), 13–16; MR **20** #3465; and Bahtin and Krasnosel'skiĭ, ibid. **123** (1958), 17–20; MR **21** #309] for a suitable class of non-linear operators. Let E be a partially ordered real Banach space, K a cone contained in the positive cone of E, and u_0 a fixed point in K. An operator A is uniformly u_0-concave if (1) $AK \subset K$, (2) $\phi_1 \leq \phi_2$ implies $A\phi_1 \leq A\phi_2$, (3) for each $\phi \in K$ there exist $\alpha, \beta > 0$ such that $\alpha u_0 \leq A\phi \leq \beta u_0$, (4) for each $\phi \in K$ such

that $\mu u_0 \leq \phi \leq \nu u_0$ ($\mu, \nu > 0$) and for any $t \in [a, b] \subset (0, 1)$ we have $At\phi \geq (1+\eta)tA\phi$, where $\eta > 0$ depends only on μ, ν, a, and b. A slight generalization is also given, together with examples. Among the theorems announced is one to the effect that if A is uniformly u_0-concave and if there exist $\phi, \psi \in K$ such that $\phi \leq A\phi$ and $\psi \geq A\psi$, then A has a fixed point in K. It is to be noted that complete continuity is not required. *G. Hufford* (Seattle, Wash.)

4367:

★**Nevanlinna, F. Über absolute Analysis.** Treizième congrès des mathématiciens scandinaves, tenu à Helsinki 18–23 août 1957, pp. 178–197. Mercators Tryckeri, Helsinki, 1958. 209 pp. (1 plate)

The first part of this paper gives an exposition, without proofs, of certain aspects of the theory of differentiation in Banach and Hilbert spaces, mentioning the mean value theorem and Stokes' theorem. The remainder is devoted to the proof of an implicit function theorem. Let z be a differentiable mapping defined on a neighborhood of the origin in $R_x \times R_y$ to R_z, where R_t is a Hilbert space. It is assumed that $z(0, 0) = 0$, that the norm of the partial derivative z_x' is bounded above, that $z_y'(0, 0)$ maps R_y one-one onto R_z and that

$$\limsup_{(x,y)\to(0,0)} |z_y'(x, y) - z_y'(0, 0)| < \mu,$$

where $0 < \mu = \inf\{|z_y'(0, 0)(k)|\,;\ k \in R_y,\ |k| = 1\}$. It is proved that there is a uniquely determined differentiable solution function y satisfying $z(x, y(x)) = 0$. Estimates are given for the size of the domain of y and it is shown that these estimates cannot be improved. The contact with F. Nevanlinna [Ann. Acad. Sci. Fenn. Ser. A.I. no. 245 (1957); MR **19**, 1067] and R. Nevanlinna [ibid. no. 185 (1955); MR **16**, 806] is very close. The author's proof requires the spaces to be Hilbert spaces, but by using some arguments of the reviewer [ibid. no 257 (1958); MR **21** #2932] a proof for Banach spaces can be obtained.

R. G. Bartle (Urbana, Ill.)

4368:

Browder, Felix E. On a generalization of the Schauder fixed point theorem. Duke Math. J. **26** (1959), 291–303.

The author treats fixed points of a completely continuous self map, f, of a Banach space X under the condition on f, suggested by the reviewer, that $f^m(X)$, $m > 1$, be bounded. The results are extended to the case of multi-valued maps. {Results valid for the finite-dimensional case of a single-valued map were obtained by the reviewer in a paper unknown to the author [Rev. Math. Pures Appl. **2** (1957), 371–374; MR **20** #1297] by somewhat analogous methods.}

D. G. Bourgin (Urbana, Ill.)

4369:

★**Michal, Aristotle D. Le calcul différentiel dans les espaces de Banach. Vol. I. Fonctions analytiques—équations intégrales.** Traduit de l'anglais par E. Mourier. Préface de M. Fréchet. Collection de monographies sur la théorie des fonctions. Gauthier-Villars, Paris, 1958. xiv + 150 pp. 3000 francs; $7.30.

This treatise gives a systematic account of the work done by A. D. Michal and his school in various fields of functional analysis. The account was made by Michal at the praiseworthy request of M. Fréchet. An almost complete manuscript in English was available at the time of the author's decease. This manuscript has been rearranged, revised, completed and rendered into French by Edith Mourier in a skillful manner. The present first volume contains basic definitions and applications to integral and differential equations and to the exponential function. Chapter I gives an elementary account of the classical Volterra theory. Chapter II contains normed linear spaces, Banach spaces, polynomials and analytic functions defined on and having values in such spaces. Here a function $f(x)$, defined in a linear normed space and having values in a B-space, is said to be analytic in a domain D if it is representable there by an absolutely and uniformly convergent series of homogeneous polynomials. In Chapter III the Fréchet differential and the Fréchet-Michal derivative are introduced and are applied to polynomials and analytic functions. The papers of Zorn in this connection are mentioned in the bibliography but not in the text. There are interesting generalizations of the theorems of Markoff and S. Bernstein on derivatives of polynomials to abstract-valued polynomials. The subject matter of Chapter IV is the differential equations (F-differential or variational equations) arising in the theory of integral equations of the Volterra type and this is extended to the Fredholm type in Chapter V. In the Volterra case, this is the differential equation of the resolvent; in the Fredholm case, in addition to the resolvent, the author considers the variational properties of the generalized Fredholm determinant considered as a functional of the kernel. In Chapter VI the dependence of the solutions of a linear differential equation $\omega'(x) = A(x)\omega(x)$, $\omega(a) = \omega_0$, on the matrix $A(x)$ is considered and expressed as an F-differential equation. In passing, a proof of the implicit function theorem for analytic functions is given based on polynomial series and majorants. This antedates a similar proof of the reviewer, now in course of publication. Finally, in Chapter VII several variational equations for the exponential function in a non-commutative ring are introduced.

This booklet contains very much interesting and valuable material and serves a useful purpose. Use of operator theory could have made the background clearer. Such use would have shown that most of the problems considered involve the existence of a resolvent and that the F-differential equations mostly are variants of one of the differential equations of the resolvent. Integration of abstract-valued functions of a complex scalar is avoided except in a few places in Chapter VI which appear to be late interpolations based on the work of this reviewer. Theorem II.25 is an obvious consequence of Cauchy's estimates once complex integration is admitted. A few inadvertencies have been noticed. Thus, Theorem II.5 on p. 23 is contradicted by the correct result on p. 25. In formula VI.2 on p. 109 the right member of the second equation should be ω_0 instead of 0. There is a reference to Naĭmark [1] on p. 27 but none in the Bibliography.

E. Hille (New Haven, Conn.)

CALCULUS OF VARIATIONS

See also 4322.

4370:

Blažek, Mikuláš. Remarks on the Noether's theorems. Mat.-Fyz. Časopis. Slovensk. Akad. Vied **8** (1958), 163–173. (Slovak. Russian and English summaries)

In mathematico-physical problems of various kinds it is important to know the behavior of higher order derivatives of wave functions possessing Lagrangians (Lagrangian density functions). E. Noether [Nachr. Gesell. Wiss. Göttingen **1918**, 235–257] established two theorems on the properties and behavior of higher order derivatives based upon the behavior and properties of the variational principle itself. The present paper deals with those theorems, providing various generalizations. There are established the conservation theorems (the invariance over a finite continuous group of transformations is required) and the identity relations between the Euler-Lagrange's expressions and their derivatives, respectively (the condition of invariance over an infinite group of transformations is satisfied). The Lagrangian is transformed following the principle of equivalence of the Euler-Lagrange equations of motion. The conditions are shown as to when one has to require the absolute or the relative invariance of the Lagrangian. Simple examples in theoretical physics of the application of both Noether theorems close the paper.

M. Z. v. Krzywoblocki (Urbana, Ill.)

4371:

Plotnikov, V. I. Differentiability of solutions of regular variational problems in non-parametric form. Mat. Sb. (N.S.) **47 (89)** (1959), 355–396. (Russian)

The main results of this paper were announced previously [Dokl. Akad. Nauk SSSR **116** (1957), 746–749; MR **20** #2646]. For the class of problems considered the variational integrand can be reduced to the form

$$F(x, y, z, p, q) = F^{(1)}(p, q) + F^{(2)}(x, y, z, p, q),$$

$$p = z_x, \quad q = z_y,$$

where $F^{(1)}$ is homogeneous of degree $\alpha \geq 2$ and $F^{(2)}$ together with its partial derivatives of orders up to 5 satisfies bounds of the form $L(p^2+q^2)^{\lambda_l}$, suitable $\lambda_l < \alpha/2$, $L > 0$. A principal lemma is the existence of a "majorizing sequence" of integrands $F_n \geq F$, F_n tending uniformly to F if (p, q) remains uniformly bounded and (x, y) in a given ε-neighborhood K_ε of a point (x_0, y_0) interior to the domain of the minimizing function $z(x, y)$, such that F_n depends only on (p, q) for p^2+q^2 large enough. Consider the minimum problem with integrand F_n and Lipschitzian boundary data on K_ε^* tending uniformly to $z(x, y)$ on K_ε^*. Any solution $z_n(x, y)$ is generalized saddle-shaped; i.e., $z_n(x, y) - ax - by$ is monotone in the sense of Lebesgue for a^2+b^2 large enough. This implies that $z_n(x, y)$ is Lipschitzian in K_{ε_1}, any $\varepsilon_1 < \varepsilon$ [Plotnikov, Uspehi Mat. Nauk **13** (1958), no. 5 (83), 191–196; MR **20** #6489]. Moreover, there is a bound

$$|\text{grad } z_n| \leq \frac{M}{\varepsilon^2 - \varepsilon_1^2}.$$

This estimate follows from the corresponding a priori bound on the gradient of solutions of certain quasilinear equations $Ar + 2Bs + Ct = D$, $r = z_{xx}$, etc., obtained using a classical device of S. N. Berstein. Then $z_n(x, y)$ tends uniformly to $z(x, y)$ in K_{ε_1} (the latter is assumed to be the unique solution in the small), from which $z(x, y)$ is Lipschitzian. By known theorems $z(x, y)$ is then of class $C^{(3)}$, analytic if F is analytic.

W. H. Fleming (Providence, R.I.)

GEOMETRY

4372:

★**Heymann, Walter. Trigonometrie der Ebene.** Fachbuchverlag Leipzig, Leipzig, 1959. 282 pp. DM 12.50.

This book, necessitated by the reorganization of the high school system in the German Democratic Republic, which replaces the former seven year pre-Technology high school by a ten year polytechnical course, treats two-dimensional trigonometry in a form suitable for the newly organized school system.

A collection of problems makes possible the use of the book for self study. Solutions of the numerous exercises are given in full. There are many clearly explained tables and diagrams in the appendix, pp. 190–280 (!).

Very useful is the last chapter, a two-page history of trigonometry. Here the author stresses its beginnings in antiquity, shows the contributions of the Arabs, and subsequently of the different European nations.

S. R. Struik (Cambridge, Mass.)

4373:

Bottema, O. On a conic section inscribed in a triangle. Nieuw Tijdschr. Wisk. **46** (1958/59), 219–224. (Dutch)

Any general triangle ABC, with circumscribed circle K_1, has a conic section K, tangent to its three sides and concentric with K_1, and $K \equiv a^4x^2 + b^4y^2 + c^4z^2 - 2b^2c^2yz - 2c^2a^2zx - 2a^2b^2xy = 0$. K is an ellipse inside [outside] of ABC for an acute- [obtuse-] angled ABC, degenerating into the double line AB for ABC rectangular at C. If p, q are the half-axes of the ellipse K, and R the radius of K_1, $p+q=R$ [resp. $p-q=R$] in an acute- [resp. obtuse-] angled ABC. K is the inscribed circle when ABC is equilateral.

S. R. Struik (Cambridge, Mass.)

4374:

Cavallaro, Vincenzo G. Geometria Brocardiana prismatica. Giorn. Mat. Battaglini (5) **6 (86)** (1958), 111–121.

The author gives still another proof of the "Teorema trigonometrica delle rotazioni" [earlier proofs by the author, Period. Mat. (3) **12** (1915), 193–209, and others] and uses it to obtain results concerning volumes of tetrahedra associated with a triangle and its Brocard points.

W. Jonsson (Winnipeg, Man.)

4375:

Demir, Hüseyin. Les cercles podaires dans le polygone inscriptible. Mathesis **68** (1959), 51–54.

$A_1, A_2, A_3, \cdots, A_n$ are n points of a circumcircle Γ. The centres of the pedal circles of a point P (not on Γ), with respect to the four triangles obtained by omitting successively a vertex of $A_{1,2,3,4}$, are on a circle, called here the pedal circle of P with respect to the quadrilateral $A_{1,2,3,4}$. Similarly, in a pentagon, the centres of the five pedal circles with respect to the five quadrilaterals lie on a circle called the pedal circle of P with respect to the pentagon; and finally the centers of the pedal circles of P with respect to the polygons of $(n-1)$ sides, obtained by omitting successively one vertex of an n-sided polygon, lie on a circle, called the pedal circle of P with respect to $A_{1,2,\ldots,n}$. Its centre, radius and other relationships to $A_{1,2,\ldots,n}$ are computed.

S. R. Struik (Cambridge, Mass.)

4376:

Lagrange, René. Sur les systèmes isogonaux de sphères. J. Math. Pures Appl. (9) **37** (1958), 225–244.

For two hyperspheres U_i ($i=1, 2$) with the centres A_i and the radii $R_i \neq 0$ (real or purely imaginary) in a Euclidean N-space E_N their (generalized) angle φ_{12} (possibly complex) is defined by

$$\cos \varphi_{12} = (2R_1R_2)^{-1}(R_1^2 + R_2^2 - \overline{A_1A_2}^2).$$

A system of n ($n>2$) hyperspheres U_i ($i=1, 2, \cdots, n$) in E_N is called strictly isogonal, if the angles φ_{ij} between each two of them are all equal: $\cos \varphi_{ij} = \cos \alpha$ for $i \neq j$. The author shows that such systems exist for $n \leq N+2$ only (with arbitrary real $\cos \alpha \neq 1$), except the case $n=N+2$ and $-(N+1)^{-1} < \cos \alpha \leq 1$, $\cos \alpha \neq 0$. All hyperspheres of the system are of the same kind (real or purely imaginary), except for $\cos \alpha = 0$. In this case, for $n=N+2$, the system consists of $N+1$ real hyperspheres and one purely imaginary hypersphere; for $n<N+2$, one, as well as none, of the hyperspheres may be purely imaginary. The author's results are in a close connexion with those obtained by the reviewer [Čas. Pěst. Mat. **81** (1956), 182–223; MR **19**, 674]. *M. Fiedler* (Prague)

4377:

Bartoš, Pavel; et Vyšín, Jan. Systèmes linéaires de similitudes directes dans le plan. Časopis Pěst. Mat. **84** (1959), 129–139. (Czech. Russian and French summaries)

Σ_n is the set of all similar maps $z' = az+b$ of the Argand plane mapping n given points $B_1, \cdots, B_n$ onto images $B_1', \cdots, B_n'$ lying, in this order, on n given lines $p_1, \cdots, p_n$. Σ_n is studied systematically for $n=2$, and some examples are given for the case $n=3$.

F. A. Behrend (Melbourne)

4378:

Green, H. G.; and Prior, L. E. On the projection of a four-point system of conics into a family of circles. Enseignement Math. (2) **5** (1959), 44–52.

"In view of the frequent use of the process of projection of two common points of a four-point system of real conics into the circular (isotropic) points with gay abandon of any difficulties which may arise from the reality or non-reality of part or parts of the consequent figure, the authors have felt the need for a closer investigation. We shall show that in order to obtain real circles when two real points are projected into the circular points two projections are necessary in general, and that only two real points, at most, on a conic remain real. We shall also establish the various types of the families of circles which are obtained." (Authors' introduction)

F. A. Behrend (Melbourne)

4379:

★**André, J. Sur quelques plans euclidiens avec un groupe de translations.** Convegno internazionale: Reticoli e geometrie proiettive, Palermo, 25–29 ottobre 1957; Messina, 30 ottobre 1957, pp. 1–2. Edito dalla Unione Matematica Italiana con il contributo del Consiglio Nazionale delle Ricerche. Edizioni Cremonese, Rome, 1958. vii+141 pp. 1800 Lire.

A "parallel structure" is a set of points and lines such that (1) two distinct points lie on one and only one line; (2) there is an equivalence relation called parallelism on lines, and through a given point there is one and only one line parallel to a given line; (3) there are three points not on a line; and (4) every line contains a given point. It is shown that if a parallel structure possesses an Abelian group of central translations, non-trivial in at least two directions, then the parallel structure and the translations may be extended simultaneously to an affine plane.

Marshall Hall, Jr. (Pasadena, Calif.)

4380:

★**Croisot, R. Sur l'irréductibilité dans les treillis géométriques et les géométries projectives.** Convegno internazionale: Reticoli e geometrie proiettive, Palermo, 25–29 ottobre 1957; Messina, 30 ottobre 1957, pp. 3–10. Edito dalla Unione Matematica Italiana con il contributo del Consiglio Nazionale delle Ricerche. Edizioni Cremonese, Rome, 1958. vii+141 pp. 1800 Lire.

Let L be a lattice with zero element; elements P, Q are called perspective, written $P \sim Q$, if for some x in L, $P \cup x = Q \cup x$ and $P \cap x = Q \cap x = 0$. The transitivity of perspectivity, i.e., (*) $P \sim Q$, $Q \sim R$ imply $P \sim R$ fails to hold, in general; however, a theorem of J. von Neumann, sharpened by the reviewer, establishes (*) in every lattice which is modular, relatively complemented and satisfies the continuity and completeness conditions for countable sets of elements [see I. Halperin, Trans. Amer. Math. Soc. **44** (1938), 537–562]. If P, Q, R are restricted to be points (i.e., atoms), then (*) holds as soon as the lattice is modular, or, more generally, if the following axioms hold (capital letters now indicate points): (A1) $P \leq Q \cup x$ implies $P \leq Q \cup X_1 \cup \cdots \cup X_m$ for some finite collection of points $\{X_i\}$ contained in x; (A2) $P \leq Q \cup x$, $P \cap x = 0$ together imply $Q \leq P \cup x$; (A3) $P \cup x \geq y \geq x$ implies $y = P \cup x$ or $y = x$. L is called a geometric lattice (treillis géométrique) if it satisfies (A1), (A2), (A3) and also: (A4) L is complete; (A5) every non-zero x in L is the union of the points it contains. In a geometric lattice an equivalence class of mutually perspective points generates an irreducible sublattice of L and L is the direct sum of all such sublattices. For L to be irreducible it is necessary and sufficient that (i) any two points are perspective; it is sufficient if (ii) every line contains at least three points or (iii) every complement of a line lies in at least three hyperplanes. It is noted that (ii) implies (iii).

In this exposé, reference is made to the work of U. Sasaki [J. Sci. Hiroshima Univ. Ser. A. **14** (1950), 100–101; MR **13**, 49] and M. Augereau [Thesis, Poitiers, 1955].

I. Halperin (Kingston, Ont.)

4381:

★**Lombardo-Radice, Lucio. Breve rassegna di problemi aperti relativi ai piani finiti.** Convegno internazionale: Reticoli e geometrie proiettive, Palermo, 25–29 ottobre 1957; Messina, 30 ottobre 1957, pp. 20–29. Edito dalla Unione Matematica Italiana con il contributo del Consiglio Nazionale delle Ricerche. Edizioni Cremonese, Rome, 1958. vii+141 pp. 1800 Lire.

A collection of problems (some well known) on finite graphic planes, with hints on possible ways for dealing with them, concerning: (i) Zorn-Levi theorem, and the search for a groupal-geometric proof of it; (ii) ovals on non-Desarguesian planes, and possible deduction of the

latter from a Galois plane by a suitable deformation; (iii) existence of finite graphic planes having certain types of projective transformations in themselves; (iv) geometric-combinatorial construction of finite graphic planes.

B. Segre (Rome)

4382:

★Zappa, Guido. Il punto di vista gruppale nello studio dei piani grafici finiti. Convegno internazionale: Reticoli e geometrie proiettive, Palermo, 25–29 ottobre 1957; Messina, 30 ottobre 1957, pp. 62–71. Edito dalla Unione Matematica Italiana con il contributo del Consiglio Nazionale delle Ricerche. Edizioni Cremonese, Rome, 1958. vii+141 pp. 1800 Lire.

This is an expository article, discussing the applications of group theory to the study of finite planes. The topics discussed include: (1) cyclic planes and the multiplier theorem, (2) planes with a regular transitive collineation group, both projective and affine planes being considered, (3) partially transitive collineations, (4) Hughes planes, (5) the Ostrom-Wagner theorem that a plane with a doubly transitive collineation group is necessarily Desarguesian.

Marshall Hall, Jr. (Pasadena, Calif.)

4383:

Ostrom, T. G. Separation, betweenness, and congruence in planes over non-ordered fields. Arch. Math. **10** (1959), 200–205.

Verf. betrachtet affine und projektive Ebenen über [kommutativen] Körpern, in denen die Quadrate in der multiplikativen Gruppe eine Untergruppe vom Index 2 bilden. Hierzu gehören also insbesondere alle endlichen desarguesschen Ebenen von Charakteristik $\neq 2$. In solchen affinen Ebenen über Galoisfeldern hatte P. Kustaanheimo eine Kongruenz und Zwischenbeziehung untersucht, die den üblichen Axiomen in abgeschwächter Form genügen [vgl. Soc. Sci. Fenn. Comment. Phys.-Math. **15** (1950), no. 19; Rend. Mat. e Appl. (5) **16** (1957), 292–296, 286–291; Math. Scand. **5** (1957), 197–201; MR **12**, 630; **20** #1268, 1269, 4808; Järnefelt und Kustaanheimo, *11te Skand. Matematikerkongress, Trondheim, 1949*, pp. 166–182, Johan Grundt Tanums Forlag, Oslo, 1952; MR **14**, 1008]. Verf. zeigt nun, daß sich diese Zwischenbeziehung in bekannter Weise aus einer Trennbeziehung der zugehörigen projektiven Ebene gewinnen läßt, und zwar sollen sich zwei Punktepaare einer Geraden genau dann trennen, wenn ihr Doppelverhältnis kein Quadrat ist. Dann trennen die Schnittpunkte einer Geraden mit einer nicht ausgearteten Quadrik jeden bezüglich der Quadrik inneren Punkt der Geraden von jedem äußeren. Hat der Koordinatenkörper noch zusätzlich die Eigenschaft, daß -1 kein Quadrat ist, so kann man in ihm einen Betrag einführen durch die Festsetzung $|x| = \pm x$, $|x|$ ist Quadrat. Für drei Punkte x_1, x_2, x_3 einer Geraden ist x_3 genau dann zwischen x_1 und x_2, wenn $|x_1-x_3|+|x_2-x_3| = \pm|x_1-x_2|$; gilt das obere Vorzeichen, so ist keiner der Punkte x_1, x_2 zwischen den beiden anderen, gilt das untere Vorzeichen, so liegt jeder der drei Punkte zwischen den beiden anderen. Schließlich wird als Abstand der beiden Punkte (x_1, y_1) und (x_2, y_2) der Wert

$$|\sqrt{|(x_1-x_2)^2+(y_1-y_2)^2|}|$$

definiert; zwei Strecken werden kongruent genannt, wenn ihre Endpunkte gleichen Abstand haben. Diese Kongruenz braucht aber nicht alle Forderungen von Kustaanheimo zu erfüllen.

H. Salzmann (Frankfurt)

4384:

Maisano, Francesco. Sulle k-calotte contenute in una quadrica rigata di $S_{3,q}$. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. **26** (1959), 35–38.

In a projective three-dimensional space over a GF(q), a k-cap is a set of k points no three of which are collinear. The present paper deals with k-caps which lie on a ruled quadric, Q, and are complete (i.e., not contained in a $(k+1)$-cap lying on Q). It is proved that, for them, either $k=2q+2$ or $k=2q+1$ and all possible types are classified.

B. Segre (Rome)

4385:

Crowe, Donald W. A regular quaternion polygon. Canad. Math. Bull. **2** (1959), 77–79.

The author considers an affine plane with a "quaternion unitary" metric, so that the distance from the origin to (x, y) is the square root of $x\bar{x}+y\bar{y}$. A line is the set of points satisfying a quaternion equation $ax+by=c$. A reflection is a periodic linear transformation which preserves distance and leaves invariant all the points on one line. A regular polygon is a configuration of points ("vertices") and lines ("edges") which is transformed into itself by a group generated by two reflections, one of which cyclically permutes the vertices on one edge while the other cyclically permutes the edges through one of these vertices. For instance, there can be an edge $x+iy=1$ containing the 4 vertices $(1, 0)$, $([1+i]/2, [-1-i]/2)$, $(0, -i)$, $([1-i]/2, [1-i]/2)$, while the first of these vertices lies on the 4 edges $x+iy=1, x+ky=1, x-iy=1, x-ky=1$. In this case there are altogether 80 vertices and 80 edges. A set of 4 defining relations is given for the group, which is found to be of order 320.

H. S. M. Coxeter (Toronto, Ont.)

4386:

Nicolas, Marie Marcel. Géométrie de la chaînette ou hypergéométrie. Z. Angew. Math. Phys. **8** (1957), 122–141.

Dall'opinione che l'Analisi e la Geometria ordinarie siano poco adatte alla trattazione di questioni tecniche inerenti a sistemi che comprendano cavi, l'A. è tratto a introdurre una geometria, che chiama "ipergeometria", nella quale, come egli dice, "le distanze siano misurate secondo catenarie". In tal senso le catenarie prendono il posto che ha la retta nella geometria ordinaria, e vengono perciò chiamate "iperrette". Sviluppando questo criterio di corrispondenza l'A. introduce altri enti (iperpiano, ipercirconferenza, iperellisse, ipernormale, iperparallele, ecc.) e ne studia talune proprietà. Si tratta, in definitiva, di proprietà delle catenarie (o di curve piane in genere), spesso sostanzialmente note anche se non sempre, a quanto è dato sapere, esplicitamente formulate; ma che, tuttavia, dalla presentazione che qui ne viene data, traggono un particolare e talvolta suggestivo rilievo. Ciò che non appare dalla Nota in esame, e sarebbe interessante, è in qual senso possa trarrsi profitto, nelle applicazioni, da questa "geometria della catenaria".

G. Tedone (Rome)

CONVEX SETS AND GEOMETRIC INEQUALITIES

See also 4247, 4401.

4387:

Schopp, J. Über eine Extremaleigenschaft des Simplex im n-dimensionalen Raum. Elem. Math. **13** (1958), 106–107.

If P is interior to a simplex A_i ($i=1, \cdots, n+1$), let A_iP intersect the face opposite A_i in B_i, and r_i be the distance from P to this face; then (1) $\prod PA_i \geqq n^{n+1} \prod PB_i$, (2) $\prod PA_i \geqq n^{n+1} \prod r_i$. Equality holds in (1) only if P is the barycenter, in (2) only if the simplex is regular.

4388:

Schopp, J. Extremaleigenschaften der Ecktransversalen des n-dimensionalen Simplex. Elem. Math. **14** (1959), 61–62.

Where A_i is a vertex of an n-dimensional simplex and P one of its inner points, A_iPB_i is the line segment intercepted by the simplex from the line through A_i and P. If $A_iP=d_i$, $PB_i=R_i$, S_k is defined to be

$$\sum_{1\leq i_1<\cdots<i_k\leq n+1} \frac{R_{i_1}\cdots R_{i_k}}{d_{i_1}\cdots d_{i_k}}.$$

It is shown that $S_k \geqq \binom{n+1}{k} n^k$ follows from a previous result of the author [review above].

D. Derry (Vancouver, B.C.)

4389:

Goldberg, M. Tetrahedra equivalent to cubes by dissection. Elem. Math. **13** (1958), 107–109.

A table is given of all known tetrahedra as described in the title, adding two to those given by Sydler, Elem. Math. **11** (1956), 78–81 [MR **18**, 61].

4390:

Klee, Victor. Some characterizations of convex polyhedra. Acta Math. **102** (1959), 79–107.

Call a (perhaps unbounded) subset K of euclidean n-space "polyhedral" if it is the intersection of finitely many closed affine halfspaces. Suppose $2\leqq j\leqq n-1$. The author proves that K is polyhedral (1) if all its j-sections are polyhedral, or (2) if K is bounded and convex and all its j-projections are polyhedral, or (3) if $j\geqq 3$ and K is closed and convex and all its j-projections have polyhedral closure, or (4) if K is a convex cone and all its j-projections are closed. These are the main results, but he establishes many refinements, and supplies counterexamples against other refinements. His last theorem deals with polyhedral approximation: For any subset X of n-space let $S(X, \varepsilon)$ be the union of open ε-balls centered at points $x\in X$, and define the "Hausdorff distance" $h(X, Y)$ between subsets X and Y to be $\inf\{d: X\subset S(Y, d) \text{ and } Y\subset S(X, d)\}$. Then for any closed convex Q, either $h(P, Q)=\infty$ for every polyhedron P, or else for every ε there is some polyhedron P with $h(P, Q)<\varepsilon$.

H. Mirkil (Hanover, N.H.)

4391:

Dvoretzky, Aryeh. A theorem on convex bodies and applications to Banach spaces. Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 223–226; erratum, 1554.

The author outlines a proof of the theorem that given $\varepsilon>0$ and a positive integer k there exists an integer N such that any symmetric bounded convex set C of dimension N contains a k-dimensional subset Y formed by the intersection of C with a Euclidean space of dimension k for which the relative frontier of Y lies between two concentric spheres whose radii are in the ratio $1+\varepsilon:1$. This result has applications to the relation between Hilbert space and more general infinite-dimensional spaces, two of which are given in the paper. The proof depends upon some quite intricate measure-theoretic arguments which are given very briefly. A slip on p. 225 has been corrected by the author in the erratum. *H. G. Eggleston* (London)

4392:

Grünbaum, B. On some covering and intersection properties in Minkowski spaces. Pacific J. Math. **9** (1959), 487–494.

Let X be a normed linear space. The expansion constant $E(X)$ of X is the greatest lower bound of positive numbers μ having the following property: For any family $\{x_i\}_{i\in I}$ of points of X and a corresponding family of non-negative numbers $\{\alpha_i\}_{i\in I}$, the relations $\|x_i-x_j\|\leqq\alpha_i+\alpha_j$ $(i, j\in I)$ imply the existence of a point $y\in X$ satisfying $\|y-x_i\|\leqq\mu\alpha_i$ $(i\in I)$. For the Euclidean n-space R^n, the author proves that $E(R^n)=(2n/(n+1))^{1/2}$. It is shown that $E(X)\leqq 2n/(n+1)$ holds for any n-dimensional space X. The case of equality is characterized. These results are similar to the known results of F. Bohnenblust [Ann. of Math. (2) **39** (1938), 301–308] and K. Leichtweiss [Math. Z. **62** (1955), 37–49; MR **16**, 1047] concerning Jung's constant of X. *Ky Fan* (Notre Dame, Ind.)

4393:

Fejes Tóth, L. Über eine Punktverteilung auf der Kugel. Acta Math. Acad. Sci. Hungar. **10** (1959), 13–19. (Russian summary, unbound insert)

Let there be given n points on the surface of the unit sphere. Denote by S_n the sum of the $\binom{n}{2}$ distances determined by their n points (the distance of two points on the sphere is defined as the shortest path connecting the two points on the surface of the sphere). The author conjectures that, for even n, $S_n\leqq\pi(n^2/4)$, with equality if and only if the points form a centrally symmetrical set. He proves this conjecture if the points are all on a great circle.

For odd n ($n=2k+1$) he conjectures that $S_n\leqq\pi k(k+1)$. He again proves this if the points are all on a great circle. He further conjectures that equality is reached if and only if the set is the union of a centrally symmetric set and a set of points $x_1, x_2, \cdots, x_{2r+1}$ on a great circle such that each halfcircle determined by x_i $(1\leqq i\leqq 2r+1)$ contains r of the points x_j in its interior.

He proves these conjectures for $n\leqq 6$ and discusses several other problems. *P. Erdös* (Budapest)

4394:

Fréchet, Maurice. Définitions de la somme et du produit par scalaire en termes de distance. Ann. Sci. École Norm. Sup. (3) **75** (1958), 223–255.

Let M denote a metric space, and Z a system $\{a, b; p, q\}$ $a, b \in M$, $p \geq 0$, $q \geq 0$, $p+q=1$. An extended mean of Z is any element c of M for which the real function $\sigma^2(x) = p\cdot(a, x)^2 + q\cdot(b, x)^2$ attains its greatest lower bound, where (r, s) denotes the distance of points r, s of M. It is shown that $\sigma^2(x) \geq p\cdot q\cdot(a, b)^2$. If $c \in M$ such that $\sigma^2(c) = p\cdot q\cdot(a, b)^2$, then it is seen that $(a, c)+(c, b)=(a, b)$ and $p\cdot(a, c) = q\cdot(b, c)$. These relations are equivalent to $(a, c) = q\cdot(a, b)$, $(b, c) = p\cdot(a, b)$ and so $(a, c)/(b, c) = q/p$. Then c is said to divide (a, b) in parts proportional to q and p. Any such point c is called a center of gravity of $Z = \{a, b; p, q\}$. Generalized centers of gravity are defined in a similar manner for systems $Z' = \{a, b; \lambda, 1-\lambda\}$, λ real. A normed linear space is called taut (tendu) provided, for each triple of its points x, y, z, $(x, y)+(y, z) = (x, z)$ if and only if there exist real numbers $\alpha \geq 0$, $\beta \geq 0$, $\alpha+\beta=1$ such that $y = \alpha x + \beta z$. It is shown that in a taut normed linear space every system Z' has a unique center of gravity and conversely, if in a normed linear space R every system Z' has a unique generalized center of gravity, then R is taut. A metric space is said to satisfy condition H provided every system Z' has a unique center of gravity. Denote such a space by M^*. The author defines neutral element, norm, sum, and product by a scalar for metric spaces M^* as follows: (1) any element of M^* is selected as the neutral element, and denoted by θ; (2) if $a \in M^*$, $\|a\| = (\theta, a)$; (3) if $x, y \in M^*$ let z denote the unique center of gravity of $\{x, y; 1/2, 1/2\}$. Then $x+y \equiv s$, where s is the generalized center of gravity of $\{\theta, z; -1, 2\}$. (4) For each real number λ and each $x \in M^*$, $\lambda\cdot x = y$, where y is the generalized center of gravity of $\{\theta, x; 1-\lambda, \lambda\}$. These notions have many of the properties of their counterparts in normed linear spaces. Certain properties are not established (e.g., associativity of addition, distributivity of scalar multiplication over addition, and the important triangle inequality for the norm). The author considers metric spaces not satisfying condition H and investigates two ways in which a "most representative" center of gravity might be defined. The concluding part of the paper gives application of the general notions of $x+y$ and $\lambda\cdot x$ to certain metric spaces.

L. M. Blumenthal (Columbia, Mo.)

GENERAL TOPOLOGY, POINT SET THEORY

See also 4105, 4106, 4109, 4368.

4395:

★Summer institute on set theoretic topology, Madison, Wis., 1955. Summary of lectures and seminars. Revised, 1958. University of Wisconsin, Madison, Wis., 1958. 157 pp.

Informal discussions, progress reports, problem lists, etc. (much of which has since undergone fuller publication). Seminar topics were: generalized manifolds; topology of 3-space; 3-dimensional manifolds; semi-metric spaces; continua and topology of the plane; fixed points; transformations and transformation groups; paracompact spaces; fibre spaces; function spaces and topological algebra; extension theorems.

4396:

Bagemihl, F. Some opaque subsets of a square. Michigan Math. J. **6** (1959), 99–103.

Let Q be a unit-square together with its interior in the Euclidean plane. A set S is called opaque if (i) $S \subset Q$, (ii) every straight line that meets Q also meets S. The distance set $\Delta = \Delta(S)$ of S is defined to be the set of all real numbers d with the property that there exist s, $s' \in S$, such that the distance between s and s' is d. The author gives some examples of opaque sets, a punctiform one and one of "finite length", following ideas of Mazurkiewicz and Denjoy. These examples might suggest that $\Delta = [0, \sqrt{2}]$ for any opaque S. However, the author derives several theorems from which it follows that this is not the case. Some problems are raised, e.g., is Δ everywhere dense in $[0, \sqrt{2}]$ for any opaque S?

J. de Groot (Lafayette, Ind.)

4397:

Povolockiĭ, A. I. Determination of the angular order of a locally simple curve. Dokl. Akad. Nauk SSSR **124** (1959), 524–526. (Russian)

Let Γ be an oriented closed curve in the plane P with a continuously turning tangent, and suppose that Γ has only a finite number of points of self-intersection or self-tangency, each of finite multiplicity. Let p denote the index of Γ, i.e., the algebraic number of complete turns that a tangent vector makes when it traverses Γ once in the positive direction. For each region D of $P-\Gamma$ denote by $q(D)$ the index of D, i.e., the algebraic number of complete turns made by a vector xy, $x \in D$, when y makes one complete positive traversal of Γ. The author shows that p is determined by the number and distribution of the regions $D_1, \cdots, D_m$ and their indices $q(D_1), \cdots, q(D_m)$. Specifically he shows that

$$p = \sum_{j=1}^{m} q(D_j) - \sum_{j=1}^{n} i(A_j),$$

where $A_1, \cdots, A_n$ are the multiple points of Γ, and

$$i(A_j) = \frac{1}{2}\left(q_1 + \frac{k-1}{k}\, q_2\right),$$

where at A_j there are $2k$ sectors $S_1, \cdots, S_{2k}$ of $P-\Gamma$ whose sides intersect at A_j, $S_i \subset D_{f(i)}$ and $q_2 = \sum_{i=1}^{2k} q(D_{f(i)})$, and $2l$ sectors $T_1, \cdots, T_{2l}$ of $P-\Gamma$ whose sides are tangent at A_j, $T_i \subset D_{f'(i)}$ and $q_1 = \sum_{i=1}^{2l} q(D_{f'(i)})$. Furthermore it is proved that $i(A_j)$ is always an integer. The problem arose from the theory of pseudo-analytic functions.

R. H. Fox (Princeton, N.J.)

4398:

Pickert, Günter. Erweiterungen eines topologischen Raumes. Arch. Math. **10** (1959), 155–161.

Let R be a topological space, of which the family of all closed sets is denoted by $\mathfrak{A}$. Let $\tilde{R} = R \cup Q$, where Q is a non-empty set disjoint from R. For each $X \subset Q$, let a family $\mathfrak{A}_X$ of closed sets of R be given. A system of six conditions concerning $\{\mathfrak{A}_X\}_{X \subset Q}$ is obtained, which is necessary and sufficient in order that

$$\tilde{\mathfrak{A}} = \{X \cup Y \mid X \subset Q,\ Y \in \mathfrak{A}_X\}$$

be the family of all closed sets of an extension space $\tilde{R}$ of R. These conditions become simpler if Q is required to be closed in $\tilde{R}$. Conditions for the compactness of $\tilde{R}$ are also given, including connections with the Alexandroff compactification. However, the author's emphasis lies in general extension spaces rather than compactifications,

so there is no mention of the related papers by P. Samuel [Trans. Amer. Math. Soc. **64** (1948), 100–132; MR **10**, 54], G. Nöbeling and H. Bauer [Math. Ann. **130** (1955), 20–45; MR **17**, 390], F. J. Wagner [Nederl. Akad. Wetensch. Proc. Ser. A. **60** (1957), 171–181; MR **19**, 436] and F. A. Behrend [Math. Z. **67** (1957), 203–210; MR **19**, 298]. In part (b) of theorem 6, "Q abgeschlossen in R" and "A_0" should be replaced by "Q abgeschlossen in $\tilde{R}$" and "$\mathfrak{A}_0$" respectively. *Ky Fan* (Notre Dame, Ind.)

4399a:

Harrold, O. G., Jr. Locally peripherally unknotted surfaces in E^3. Ann. of Math. (2) **69** (1959), 276–290.

4399b:

Griffith, H. C. A characterization of tame surfaces in three space. Ann. of Math. (2) **69** (1959), 291–308.

Each of these papers gives a new local characterization of tame surfaces in Euclidean 3-space, R^3. Each of them uses the semi-linear surface approximation theorem of Bing [same Ann. **65** (1957), 456–483; MR **19**, 300]. Griffith uses the theorem of Bing [ibid. **59** (1954), 145–158; MR **15**, 816] and Moise [ibid. **59** (1954) 159–170; MR **15**, 889] that locally tame surfaces are tame. Harrold presents alternate arguments; one uses the characterization of Griffith and the other uses the Bing-Moise theorem referred to above. For a k-manifold M in R^n, local peripheral unknottedness (LPU) is defined inductively on k as follows: 0-cells and 0-spheres are LPU; M is LPU at $x \in M$ if, for each $\epsilon > 0$, there is a closed n-cell of diameter less than ϵ whose interior contains x and whose boundary meets M in an LPU cell or sphere, according as x lies on the boundary of M or not. A topological 2-cell C in R^3 is said to have the disk property if for each arc $T \subset C$ which intersects the boundary of C, ∂C, only in its end points, and $\epsilon > 0$, there is a topological 2-cell D in R^3 such that $(\partial D) \cap C = \emptyset$, $D \cap C$ is an arc in C intersecting ∂C only in its end points and lying at a Hausdorff distance from T of less than ϵ, and there is an $\eta > 0$ such that any connected set in R^3 of diameter less than η which meets both components of $C - C \cap D$ also meets D. If every sub-2-cell of C has the disk property, then C is said to have the hereditary disk property. A topological 2-cell C has the enclosure property if each neighborhood of C contains a 2-sphere which contains C in its bounded complementary domain. C has the strong enclosure property if for each pair U and V of open sets in ∂C, there is a pair of disjoint topological rays each meeting C only in their initial points, one in U and the other in V, and each neighborhood of C contains a 2-sphere with C in its bounded complementary domain which meets each of the rays in a single point. These various properties are invariant under homeomorphisms of R^3 onto itself.

Griffith shows that a topological 2-cell in R^3 is tame if it has the strong enclosure property and the hereditary disk property. Then by the Bing-Moise theorem, a 2-manifold without boundary is tame if each of its points lies in the interior of a 2-cell in the manifold having those two properties. A slight variant also characterizes the tame 2-manifolds with boundary. Harrold shows that a topological 2-cell in R^3 which is LPU satisfies the conditions of Griffith. He then gives a more direct proof that a closed 2-manifold in R^3 which is LPU at each of its points is tamely imbedded. *E. Dyer* (Chicago, Ill.)

4400:

Jones, F. B.; and Young, G. S. Product spaces in n-manifolds. Proc. Amer. Math. Soc. **10** (1959), 307–308.

Combining several known facts from point set topology and dimension theory the following theorem is established: Let $A_1, \cdots, A_n$ be nondegenerate locally compact connected metric spaces. If $A_1 \times \cdots \times A_n$ can be topologically imbedded in an n-manifold, then each A_i is either an arc, a simple closed curve, a ray or an open curve. The result is valid for generalized manifolds too.

S. Mardešić (Zagreb)

4401:

★**Berge, Claude. Espaces topologiques: Fonctions multivoques.** Collection Universitaire de Mathématiques, Vol. III. Dunod, Paris, 1959. xi + 272 pp. 3400 francs.

This is an introduction to set theory, general topology and topological vector spaces. According to the author, the book is addressed to students of pure mathematics as well as those of applied mathematics. While the choice of certain topics is motivated by their use in theory of graphs, theory of games and mathematical economics, most topics are included without such application in mind (e.g., Hilbert's example of Peano curves). The book can be understood by a reader without any specific prerequisite and should be easy reading. The exposition is fairly discursive and proceeds at a leisurely pace. Sometimes a theorem is proved in two places, first in a special case, then in the general form. Although a great part of the contents is standard, there are exceptions where new concepts and recent results are presented.

The first three chapters deal with set theory. Chapter 1 (operations of sets, filter bases, lattices of sets, limits superior and inferior of a family of sets) and Chapter 3 (ordered sets, cardinal and ordinal numbers, Zorn's lemma) are standard. Chapter 2 discusses mappings (single-valued and multi-valued) from a set into another, their operations, and the upper and lower inverses of a multi-valued mapping.

General topology is treated in the next three chapters. Chapters 4 (basic definitions and elementary theorems on topological spaces) and 5 (topology of metric spaces) follow the usual and rather classical expositions. Chapter 6 deals with multi-valued mappings from a topological space into another. Two kinds of semi-continuity are defined and studied. The material of Chapters 2 and 6 is originated from theory of games. Its systematic development has begun in earlier publications of the author [J. Math. Pures Appl. (9) **32** (1953), 129–184; *Théorie générale des jeux à n personnes*, Mémor. Sci. Math. no. 138, Gauthier-Villars, Paris, 1957; MR **16**, 500; **20** #5700].

Chapter 7 introduces real vector spaces, convex sets and multi-valued linear mappings. Chapter 8 is restricted to Euclidean n-space R^n and discusses the following topics: separation of convex sets, the Krein-Milman theorem (in R^n), the Knaster-Kuratowski-Mazurkiewicz theorem on simplexes, Kakutani's generalization of the Brouwer fixed-point theorem, G. Birkhoff's theorem [Univ. Nac. Tucumán. Revista A. **5** (1946), 147–151; MR **8**, 561] on extreme points of the set of doubly stochastic matrices and a related theorem of Hardy-Littlewood-Pólya, Helly's

theorem on intersection of convex sets, minimax theorem, convex functions and generalizations, i.e., sub-ϕ functions in the sense of E. F. Beckenbach [Bull. Amer. Math. Soc. **43** (1937), 363–371] and S-convex functions introduced by I. Schur [S.-B. Berlin. Math. Ges. **22** (1923), 9–20] and studied more recently by A. Ostrowski [J. Math. Pures Appl. (9) **31** (1952), 253–292; MR **14**, 625]. In Chapter 9 on topological vector spaces, the main theorems are: separation of convex sets, Krein-Milman theorem, generalization of Kakutani's fixed point theorem to locally convex topological vector spaces, Alaoglu's compactness theorem. Here separation of convex sets by convex functions is treated as in the author's previous paper [Bull. Soc. Math. France **82** (1954), 301–315; MR **16**, 717] and is based on the notion of regular convexity with respect to a family of lower semi-continuous convex functions.

The reviewer noted a few defects of exposition and inaccuracies as follows. The notation for a function is sometimes $f(x)$ instead of f. A family of sets is almost always considered as an indexed family, $\mathfrak{A}=(A_i \mid i \in I)$, even when there is no need to do so. A square matrix of order n is denoted by a_y^x, which is unnecessarily cumbersome. In the theorem on p. 42, condition (1) can be weakened to $A \in \mathfrak{A}$ or even replaced by the requirement that $\mathfrak{A}$ is non-empty. On p. 4, the surface torus is called "tore à trois dimensions". On p. 107, the graph for the function g is wrong. In theorem 1 on p. 244, for the part concerning the openness of λG, one should assume $\lambda \neq 0$. The book is marred by misprints, which are too numerous to be corrected here. The most conspicuous ones are: Beckerbach (p. 167), Birckhoff (p. 189) and Hilly (p. 208).

Ky Fan (Notre Dame, Ind.)

4402:

de Groot, J.; and de Vries, H. Metrization of a set which is mapped into itself. Quart. J. Math. Oxford Ser. (2) **9** (1958), 144–148.

This paper continues the study of the problem [Ellis, Quart. J. Math. Oxford Ser. (2) **4** (1953), 117–119; MR **15**, 51; Powderly and Tong, ibid. **7** (1956), 1–2; MR **18**, 916]: given a set S and a mapping ϕ of S into itself, how can one construct a topology in S, such that the topology is in general non-trivial and renders ϕ continuous? The following results are obtained. If S is an infinite set, one can always introduce a non-discrete metric in S under which ϕ is continuous. If S has at most continuously many elements, then S can be embedded into the Cantor discontinuum so that ϕ becomes continuous. If ϕ is one to one, then the ϕ in the above theorems can be made to be a topological map. Other related results are also given.

Hing Tong (Middletown, Conn.)

4403:

Cheng, C. M. On orbital topologies. Advancement in Math. **2** (1956), 663–666. (Chinese)

The results of this paper are related to the work of Ellis [Quart. J. Math. Oxford Ser. (2) **4** (1953), 117–119; MR **15**, 51]. Let S be a set. For every $x \in S$ there exists $D(x) \subset S$ such that $x \in D(x)$, and $y \in D(x) \Rightarrow D(x) \supseteq D(y)$. Let $O(x)=\{x\} \cup (D(x)-A)$, where A is a finite set. These $O(x)$ define a neighborhood system which makes S a topological space S_D. The following results, among others, are obtained: S_D is a T_1 topological space. S_D is a T_2 space if and only if $x \neq y \Rightarrow D(x) \cap D(y)$ is a finite set. If δ is a mapping of S into S such that (i) $\delta D(x) \subseteq D(\delta x)$ for every $x \in S$ and (ii) δ is finite to one, then δ is continuous over S_D. If, for every $x \in S$, $D_1(x) \supseteq D_2(x)$, then every closed set of S_{D_1} is also a closed set of S_{D_2}.

Hing Tong (Middletown, Conn.)

4404:

Balachandran, V. K. On the lattice of convergence topologies. J. Madras Univ. Sect. B **28** (1958), 129–146.

Let R be an abstract set. A class of sequences in R to each of which has been associated an abstract limit point in R, in symbols $(x_n) \to x$, is called a convergence topology t if the following postulates are satisfied: (P_1) If $x_n = x$ for all n, then $(x_n) \to y$ belongs to t if and only if $y = x$; (P_2) if $(x_n) \to x$ belongs to t, then $(x_{k_n}) \to y$ belongs to t if and only if $y = x$. The collection of all such topologies forms a complete lattice L_0 which is isomorphic to a complete ring of sets. Lattice properties of L_0 are investigated as well as lattice properties of L_1, L_2, L_3, L_4 which are obtained from L_0 by imposing increasingly stronger conditions on the topologies. In particular it is shown that L_4, the lattice of topologies for which $(x_n) \nrightarrow x$ implies there is a subsequence none of whose subsequences converge to x, is a Boolean algebra and, in fact, consists of those elements of L_0 which are pseudo-complements of pseudo-complements.

R. P. Dilworth (Pasadena, Calif.)

4405:

Glicksberg, Irving. Stone-Čech compactifications of products. Trans. Amer. Math. Soc. **90** (1959), 369–382.

Let $\beta(X)$ denote the Stone-Čech compactification of the completely regular (Hausdorff) space X. Any product $\mathbf{P}X_\alpha$ of completely regular spaces X_α is embedded in a natural way as a dense subspace of the compact space $\mathbf{P}\beta(X_\alpha)$. The question thus arises as to whether the latter space is the Stone-Čech compactification of the subspace (written $\beta(\mathbf{P}X_\alpha) = \mathbf{P}\beta(X_\alpha)$). The erroneous statement that $\beta(\mathbf{P}X_\alpha) = \mathbf{P}\beta(X_\alpha)$ for every product $\mathbf{P}X_\alpha$ appeared in Hewitt, Trans. Amer. Math. Soc. **64** (1948), 45–99 [MR **10**, 126]. The main result of the present paper is that $\beta(\mathbf{P}X_\alpha) = \mathbf{P}\beta(X_\alpha)$ if and only if either (1) $\mathbf{P}_{\alpha \neq \alpha_0} X_\alpha$ is a finite set for some α_0, or (2) $\mathbf{P}X_\alpha$ is pseudo-compact (which requires more than pseudo-compactness of each X_α). Condition (1) is, of course, a trivial case that can be disposed of immediately. The necessity of the conditions, and some partial results in the converse direction, were obtained independently by M. Henriksen and J. R. Isbell [abstract, Bull. Amer. Math. Soc. **63** (1957), 145–146]. In the proof of the sufficiency of (2), the author utilizes two characterizations of pseudo-compact spaces obtained by him in Duke Math. J. **19** (1952), 253–261 [MR **14**, 288]: they are the spaces on which Ascoli's theorem holds, and the spaces in which every sequence of nonvoid, pairwise disjoint open sets has a cluster point. Part of the proof of the sufficiency also yields the following theorem. Let $\{X_\alpha\}$ be a set of uncountably compact Hausdorff spaces, each having at least two points. For $b \in \mathbf{P}X_\alpha$ let X^b be the proper subspace of the product consisting of all points $x \in \mathbf{P}X_\alpha$ with $x_\alpha \neq b_\alpha$ for at most countably many α. Then $\mathbf{P}X_\alpha = \beta(X^b)$.

The main theorem is then used to prove that if X and Y are pseudo-compact, X is locally $\aleph_\gamma$ compact, and each non-P-point of Y [L. Gillman and M. Henriksen, Trans. Amer. Math. Soc. **77** (1954), 340–362; MR **16**, 156] has a base of neighborhoods of cardinality $\leq \aleph_\gamma$, then $X \times Y$ is

pseudo-compact. This result, in turn, is used together with a diagonal process to prove the following theorem. Let $\{X_\alpha\}$ be a set of pseudo-compact spaces. Then $\mathbf{P}X_\alpha$ is pseudo-compact in each of the following cases: (a) all but one X_α is locally compact; (b) for all but one α, each non-P-point of X_α is a G_δ; (c) for every α, X_α is $\aleph_\gamma$ compact and each non-P-point of X_α has a base of neighborhoods of cardinality $\leq \aleph_\gamma$.

Several other items related to the theorems quoted here are also discussed.

{The author notes that in footnote 7, lines 6–7 should be replaced by "$x \in X$ implies $Y^x \subset X$ so that $\beta(X) = Y$."; and in line 9, "Y" should be replaced by "$\beta(X) \setminus X$" (twice).} *C. W. Kohls* (Urbana, Ill.)

4406:

Michael, E. Yet another note on paracompact spaces. Proc. Amer. Math. Soc. **10** (1959), 309–314.

If $\mathfrak{U}$ and $\mathfrak{V}$ are collections of subsets of a Hausdorff space X, then $\mathfrak{V}$ is cushioned in $\mathfrak{U}$ if one can assign to each $V \subset \mathfrak{V}$ a $U_V \subset \mathfrak{U}$ such that for every subset $\mathfrak{V}' \subset \mathfrak{V}$

$$(\bigcup\{V \mid V \in \mathfrak{V}'\})^- \subset \bigcup\{U_V \mid V \in \mathfrak{V}'\}.$$

A refinement of $\mathfrak{U}$ which is cushioned in $\mathfrak{U}$ is called a cushioned refinement of $\mathfrak{U}$. A σ-cushioned refinement is now defined as usual (the author forgets to state in his definition that it has to be a refinement).

Theorem: A T_1-space X is paracompact if and only if every open covering of X has a cushioned refinement or more generally an open σ-cushioned refinement. This theorem is applied to obtain an alternate proof of a metrization theorem of J. Nagata.

J. de Groot (Lafayette, Ind.)

4407:

Isbell, J. R. On finite-dimensional uniform spaces. Pacific J. Math. **9** (1959), 107–121.

It is proved that every finite-dimensional uniform complex [cf. J. R. Isbell, same J. **8** (1958), 67–86, 941; MR **20** #4261] is a uniform absolute neighborhood retract; on the other hand, the real line is never (i.e., under no uniformity consistent with its topology) a uniform absolute retract. Dimensions δd and Δd of a uniform space μX are considered (δd is the least n such that every finite uniform covering has an n-dimensional uniform refinement; for Δd, omit "finite"). It is shown that, for any μX, either $\delta d(\mu X) = \Delta d(\mu X)$ or $\Delta d(\mu X) = \infty$, and it is stated that, for $n = 0, 1, 2, \cdots, \infty$, there is μX with $\delta d = 0$, $\Delta d = \infty$; for $n = 0$, an example is given where, moreover, μX is countable and (topologically) discrete. Finally, a considerably simplified proof is given of Smirnov's [Mat. Sb. (N.S.) **40** (**82**) (1956), 137–156; MR **19**, 300] characterization of sets $S \subset E^n$ with $\delta dS = n$. *M. Katětov* (Prague)

4408:

Takasaki, Tatsumi. Some properties of uniformisable space. Bull. Fukuoka Gakugei Univ. III **8** (1958), 5–12. (Japanese)

The author gives proofs of Exercises (5), (6), (7), (8) in N. Bourbaki, *Topologie générale*, chap. IX [Actual. Sci. Ind. no 1045, Hermann, Paris, 1948; MR **10**, 260], § 1. Some of them are not correct. *H. Nakano* (Sapporo)

4409:

Sierpiński, W. On a certain definition of a complete space. Wiadom. Mat. (2) **1** (1955/56), 206–207. (Polish)

Excerpt from Ganita **1** (1950), 13–16 (in French) [MR **12**, 729].

4410:

Stone, A. H. Metrisability of unions of spaces. Proc. Amer. Math. Soc. **10** (1959), 361–366.

In 1956, Yu. Smirnov [Fund. Math. **43** (1956), 387–393; MR **18**, 813] proved: (Y) If the locally countably compact Hausdorff space S is the union of countably many separable metric subsets S_n, then S is metrizable. The author obtains some analogous results, not assuming separability of the S_n. The following are typical. (1) If the collectionwise normal, locally countably compact space S is the union of countably many closed metrizable subsets S_n, then S is metrizable. (2) If the regular space S is the union of countably many open metrizable subsets with compact frontiers, then S is metrizable.

{Reviewer's comment: Both Smirnov's proof of (Y) and the author's proof of (1) are quite complicated. The following simpler proofs are due to H. Corson:

First prove (Y) for compact S: Since $S \times S$ is the union of countably many separable metric spaces, it is hereditarily Lindelöf, and hence every closed set—in particular the diagonal—is a G_δ; therefore S (being compact) is metrizable. To prove (Y) in general, observe first (with Smirnov) that S is hereditarily Lindelöf, and hence locally compact. Thus S is (a) locally metrizable (by (Y) for compact spaces), and (b) regular and therefore (being Lindelöf) paracompact; hence S is metrizable.

To prove (1), it suffices to show that S is paracompact, since it is locally metrizable by (Y). But if $\mathscr{U}$ is an open covering of S, then for each n the covering $\{U \cap S_n \mid U \in \mathscr{U}\}$ of S_n has a σ-discrete closed refinement (since S_n is paracompact), hence $\mathscr{U}$ has a σ-discrete open refinement (by collectionwise normality); this implies that S is paracompact.} *E. Michael* (Seattle, Wash.)

4411:

Arin', È. I.; and Zander, M. K. Partially uniform convergence. Latvijas PSR Zinātņu Akad. Vēstis **1959**, no. 6 (143), 75–82. (Russian)

Let E be a compact metric space, and $\mathfrak{M}$ a family of closed subsets of E such that: (1) $\bigcup\{M : M \in \mathfrak{M}\} = E$; (2) for every $M_0 \in \mathfrak{M}$ and $\varepsilon > 0$ and $x_0 \in M_0$, there is a $\delta > 0$ and an $M' \in \mathfrak{M}$ such that if $M \in \mathfrak{M}$ and $M \cap U_\delta(x_0) \neq \emptyset$, then $(M \cap M_0 \cap U_\varepsilon(x_0)) \cup (M \cap M' \cap U_\varepsilon(x_0)) \neq \emptyset$. Let f be a real-valued function on E that is continuous on each set $M \in \mathfrak{M}$. Then there is a sequence of continuous real-valued functions f_n on E such that $\lim_{n\to\infty} f_n(x) = f(x)$, uniformly on each set $M \in \mathfrak{M}$. This generalizes a classical theorem of R. Baire for functions on the plane that are continuous in each variable separately.

E. Hewitt (Seattle, Wash.)

4412:

Gluskin, L. M. Semigroups of topological mappings. Dokl. Akad. Nauk SSSR **125** (1959), 699–702. (Russian)

Let Ω be a closed bounded set in n-dimensional Euclidean space with a non-empty interior; let S be the semigroup of all topological mappings of Ω into itself.

The components of the interior of Ω are indexed as Ω_i. If $a \in S$, the rank of a is the power of the set of those indices i such that $a\Omega \cap \Omega_i$ is not empty. Let S_r^β be the set of all those mappings a in S with rank not exceeding r and with the property that, if $a\Omega \cap \Omega_i$ is not empty, Ω_i contains a closed β-connected subset whose interior contains $a\Omega \cap \Omega_i$. The paper studies these sets S_r^β, all of which are right ideals of S. A principal theorem characterizes the automorphisms of S_r^β. A certain regular equivalence relation σ is defined on S. The factor-semigroup S/σ is a semigroup with kernel S_1^1/σ isomorphic to a completely simple semigroup K without zero; furthermore K is a topological invariant of the set Ω. Several additional results are obtained in the cases when the dimension n is 1 or 2.

R. A. Good (College Park, Md.)

4413:

Suzuki, Jingoro. Note on a theorem for dimension. Proc. Japan Acad. **35** (1959), 201–202.

The author proves the following. Let f be a closed continuous map of a metric space X onto a topological space Y. Then: (1) If for every $y \in Y$, $\operatorname{card} f^{-1}(y) = k < \aleph_0$, then $\dim X = \dim Y$; (2) if for every $y \in Y$, $\operatorname{card} f^{-1}(y) < \aleph_0$, then for each finite m,

$$\dim\{y \mid \operatorname{card} f^{-1}(y) = m\} \leq \dim X.$$

Theorem (1), and its immediate corollary, (2), strengthen a result of Nagami [same Proc. **34** (1958), 503–506; MR **21** #862].

J. Dugundji (Los Angeles, Calif.)

ALGEBRAIC TOPOLOGY

4414:

Coxeter, H. S. M. The four-color map problem, 1840–1890. Math. Teacher **52** (1959), 283–289.

A historical review of matters connected with early attempts to solve the four-color problem. The author (quite correctly) gives an example, using a map of only nine regions, to point out the fallacy in the Kempe "proof" of 1879. This map, however, naturally has at least one region with fewer than five sides; so the Kempe treatment would not require the reduction of a pentagon. The simplest such example known to the reviewer, using a map containing no region of fewer than five sides, is due to Errera, who used a map consisting of a pentagon surrounded by three rings of five regions each plus an exterior pentagon, making 17 regions in all. The original example of Heawood used a slightly larger map of considerably less symmetry.

D. C. Lewis, Jr. (Baltimore, Md.)

4415:

Mardešić, Sibe. Comparison of singular and Čech homology in locally connected spaces. Michigan Math. J. **6** (1959), 151–166.

For any space X, let $H(X;C)$ = Čech and $H(X;S)$ = singular homology, let

$$\nu^q : H^q(X;C) \to H^q(X;S), \quad \nu_q : H_q(X;S) \to H_q(X;C)$$

denote the natural homomorphisms, and let lc_S denote homology local connectedness using singular homology. The author proves: (a) If X is paracompact Hausdorff, lc_S^p and semi-$(p+1)$-lc_S then ν^q, ν_q are isomorphisms (onto) for all $q \leq p+1$; if the semi-$(p+1)$-lc_S condition alone is dropped, ν^{p+1} is a monomorphism, but an example shows that ν_{p+1} need not be an epimorphism. (b) If X is locally compact Hausdorff and lc_S^p, then $H_q(X;S) \approx H_q(X;$ Čech on compact carriers) for all $q \leq p$.

J. Dugundji (Los Angeles, Calif.)

4416:

★**Peterson, Franklin P. Functional higher order cohomology operations.** Symposium internacional de topología algebraica [International symposium on algebraic topology], pp. 159–164. Universidad Nacional Autónoma de México and UNESCO, Mexico City, 1958. xiv + 334 pp.

Summary of results of #4417 below.

4417:

Peterson, Franklin P. Functional cohomology operations. Trans. Amer. Math. Soc. **86** (1957), 197–211.

This paper is devoted to the study of stable cohomology operations. Let $\mathfrak{P}$ be a Postnikov system in which all the k-invariants are suspensions (this is the assumption of stability). $\mathfrak{P}$ can be extended in a natural way so as to become an "exact couple" of spaces in the sense that applying the functor $\pi(K, \cdot)$ gives an exact couple in the usual sense. The boundary operators in the derived couples are then stable cohomology operations. The author gives two definitions of the notion of functional cohomology operation (in the stable case). One of these is a generalization of Steenrod's original treatment of primary operations and the other is new. It is then proved that up to a natural automorphism these definitions coincide. [It has been shown more recently that this automorphism is the identity; cf. Peterson and Stein, Amer. J. Math. **81** (1959), 281–305.]

The author next proves the following result. Let L be a finite-dimensional CW-complex and let K be a loop-space (or let $\dim L \leq 2n-2$ and let K be $(n-1)$-connected). Let $a : L \to K$ be a map such that, for any group G, $a^* : H^*(K;G) \to H^*(L;G)$ is trivial. Then a is homotopic to a constant if and only if all functional cohomology operations corresponding to a and an operation in the Postnikov system of L are zero. (Each k-invariant of L gives rise to a cohomology operation and it is the functional operations corresponding to these and to a that occur here.)

Finally the author gives a sufficient cohomological condition to conclude that the composition $a \circ b$ of $a : L \to K$ and $b : M \to L$ is non-trivial and using this he gives a new proof for some well-known results of Adem. The condition is too complicated to state here.

N. Stein (New York, N.Y.)

4418:

Peterson, Franklin P.; and Thomas, Emery. A note on non-stable cohomology operations. Bol. Soc. Mat. Mexicana (2) **3** (1958), 13–18.

θ being a primary cohomology operation of type $(\pi, n; G, q)$ and f a continuous map (which may be assumed to be an inclusion) of a space Y into a space X, N. E. Steenrod defined [Ann. of Math. (2) **50** (1949), 954–988); MR **11**, 122] a functional cohomology operation θ_f. When θ is additive, it is obvious that θ_f maps on cosets of $H^*(Y;G)$ modulo a subgroup $L(\theta, f)$. The authors prove

that this is true even if additivity holds only for elements of the form $v+\delta w$ (where δ denotes the coboundary operator of the cohomology sequence of the pair X, Y). They prove this restricted additivity by considering the binary operation $y_\theta \vdash (x_1, x_2)$ measuring the deviation from additivity, introduced by Eilenberg and MacLane [ibid. **60** (1954), 513–557; MR **16**, 392] and using the property $y_\theta \vdash (x, 0)=0$.

A second proof is given starting from the definition of functional cohomology operations given by F. Peterson [#4417 above], using a Postnikov system for a space Z with the two non-vanishing homotopy groups $\pi_n(Z)=\pi$ and $\pi_{q-1}(Z)=G$. The group structure of the images by θ_f is not obvious when Z may not be considered as a space of loops. But it is shown here that the fibre space of the Postnikov system is a principal fibre space with the monoid $K(G, q-1)$ as fibre (because it is induced from θ and the path space over a $K(G, q)$), and the property of θ_f results from a lemma on mappings of a space into a principal fibre space. *G. Hirsch* (Brussels)

4419:

Araki, Shôrô. Steenrod reduced powers in the spectral sequences associated with a fibering. I, II. Mem. Fac. Sci. Kyusyu Univ. Ser. A. Math. **11** (1957), 15–64, 81–97.

Problem six in Massey, Ann. of Math. (2) **62** (1955), 327–359 [MR **17**, 653] is (vaguely) to put the Steenrod squaring operations and reduced pth power operations into the cohomology spectral sequence of a fibre space. In these two papers the author solves this problem [it was solved independently by R. Vázquez, Bol. Soc. Mat. Mexicana (2) **2** (1957), 1–8; MR **19**, 973]. Let (E, π, B) be a fibre space with fibre F. Let $E_r^{a,b}$ be the cohomology spectral sequence of this fibre space with coefficients mod p. Then the author defines operations, for p an odd prime,

$$
\begin{aligned}
&{}_P\mathscr{P}_p^s : E_r^{a,b} \to E_r^{a,b+2s(p-1)} && \text{for } 1 \leqq r \leqq \infty,\\
&{}_B\mathscr{P}_p^s : E_r^{a,b} \to E_{p(r-2)+2}^{a+(2s-b)(p-1),pb} && \text{for } 2 \leqq r \leqq \infty,\\
&{}_P\mathrm{Sq}^j : E_r^{a,b} \to E_r^{a,b+j} && \text{for } 1 \leqq r \leqq \infty,\\
&{}_B\mathrm{Sq}^j : E_r^{a,b} \to E_{2r-2}^{a+j-b,2b} && \text{for } 2 \leqq r \leqq \infty.
\end{aligned}
$$

The author proves many properties of these operations; e.g., if $r=2$, and $b=0$, then ${}_B\mathscr{P}_p^s$ is the ordinary Steenrod reduced power operation in the cohomology of B; or if $r=2$ and $a=0$, then ${}_F\mathscr{P}_p^s$ is the ordinary Steenrod reduced power operation in the cohomology of F, and furthermore if $2s=b$, then ${}_B\mathscr{P}_p^s$ is the pth power cup product in the cohomology of F. Finally, the author proves the analogs of the Cartan formula and the Adem relations for his operations. *F. P. Peterson* (Cambridge, Mass.)

4420a:

Toda, Hirosi. p-primary components of homotopy groups. I. Exact sequences in Steenrod algebra. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **31** (1958), 129–142.

4420b:

Toda, Hirosi. p-primary components of homotopy groups. II. Mod p Hopf invariant. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **31** (1958), 143–160.

4420c:

Toda, Hirosi. p-primary components of homotopy groups. III. Stable groups of the sphere. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **31** (1958), 191–210.

In 1952 the author computed the homotopy groups $\pi_{n+r}(S^n)$ for $r \leqq 8$ [for details see J. Inst. Polytech. Osaka City Univ. Ser. A. Math. **3** (1952), 43–82; MR **15**, 732]. Later he published without proof tables listing these groups for the cases $9 \leqq r \leqq 13$ [see C. R. Acad. Sci. Paris **240** (1955), 147–149; MR **16**, 847] and for the cases $r=14$ or 15 [C. R. Acad. Sci. Paris **241** (1955), 849–850; MR **17**, 289]. However, after publication of these tables, it became apparent that the structure given for the 3-primary component of $\pi_{n+10}(S^n)$ and $\pi_{n+11}(S^n)$ was erroneous. For example, when the results in this table were used to compute the 10-dimensional homotopy groups of the classical Lie groups [see the table given by A. Borel on p. 428 of Bull. Amer. Math. Soc. **61** (1955), 397–432; MR **17**, 282] the result obtained contradicted the periodicity theorems of R. Bott for the stable homotopy of the classical groups [Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 933–935; MR **21** #1588].

Apparently the present sequence of three papers is part of the author's plan to publish the complete details of his computations of the homotopy groups of spheres, including the correction of the above-mentioned error. He has already published the details of his computations of the 2-primary component of the stable groups of the r-stem for $r \leqq 13$ [Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **31** (1958), 33–64; MR **20** #7263].

The main result of these three papers gives the structure of the p-primary component of the stable groups of the k-stem $\pi_{n+k}(S^n)$ for all values of $k<2p^2(p-1)-3$ where p is any odd prime. For this range of values of k, the most complicated p-primary component occurring is the direct sum of a cyclic group p and a cyclic group of order p^2, and that occurs only once. A cyclic group of order p^2 occurs for $p-2$ different values of k, and of course a cyclic group of order p occurs more frequently.

The main results of part I are propositions 1.5 and 1.6 which state that certain sequences of left modules and homomorphisms over the Steenrod algebra mod p are exact, and proposition 1.7 which gives the structure of the kernel of a certain homomorphism of left modules over the Steenrod algebra. The precise statements are too complicated to give here. However, these results are used in an essential way in the computations in part III.

The mod p Hopf homomorphism $H_p: \pi_{m+2t(p-1)-1}(S^m) \to Z_p$ for an odd prime p is an obvious generalization of the classical Hopf invariant. By definition, it is a homomorphism onto if and only if there exists a cell complex $K=S^m \cup e^{m+2t(p-1)}$ in which the Steenrod reduced pth power

$$\mathscr{P}^t : H^m(K, Z_p) \to H^{m+2t(p-1)}(K, Z_p)$$

is an isomorphism. It follows readily from Adem's relations on Steenrod pth powers that $H_p=0$ unless $t=p^j$ for some integer $j \geqq 0$. The main result of part II asserts that if $H_p \neq 0$ for $t=p^r$, then $H_p=0$ for $t=p^{r+1}$; i.e., H_p cannot be non-trivial for two consecutive powers of p. Since it is known that $H_p \neq 0$ for $t=1$, it follows that $H_p=0$ for $t=p$. This result is crucial in the calculations of part III; it makes it possible to decide the structure of the p-primary component of $\pi_{n+2p(p-1)-2}(S^n)$. It was at just this point that the error in the author's previously published tables occurred (for $p=3$).

This main result about the mod p Hopf homomorphism is analogous to a theorem previously proved by J. F. Adams about the classical mod 2 Hopf homomorphism [C. R. Acad. Sci. **245** (1957), 24–25; MR **19**, 570]. The proof given by Toda for the case of an odd prime p is quite complicated. Among other things, it makes use of the results in part I. It is apparently an open question whether there exist any values of $t>1$ for which

$$H_p:\pi_{n+2t(p-1)-1}(S^n)\to Z_p$$

is non-trivial.

In part III the actual calculations of the p-primary components of the stable homotopy groups of spheres are carried out. The method used is the calculation of the mod p cohomology of the Postnikov system of a high dimensional sphere. Since all computations are made in the stable range, the spectral sequences of the various fibre spaces involved all reduce to exact sequences. These exact sequences are sequences of left modules over the Steenrod algebra mod p, and the results of part I are used over and over again. It is also necessary to compute the various higher order Bockstein operators for each space in the Postnikov system in order to determine from the mod p cohomology whether or not there are elements of order p^2, p^3, etc., in the integral homology. The whole procedure is analogous to that which the author has used previously to compute the 2-primary components of the stable homotopy groups of spheres [in the paper already mentioned]. The exposition of the actual calculations is given in a form which is valid for any odd prime p. Naturally this makes it much more complicated than if it were given for the case of a single prime p (e.g. $p=3$).

W. S. Massey (Providence, R.I.)

4421:

Schubert, Horst. Semisimpliziale Komplexe. Jber. Deutsch. Math. Verein. **61** (1958), Abt. 1, 126–138.

Expository lecture on the basic facts about semisimplicial complexes, leading up to the algebraic characterization of homotopy type in the category of CW-complexes by means of Postnikov complexes (i.e., minimal Kan complexes). *J. A. Zilber* (Providence, R.I.)

4422:

Cockcroft, W. H. The cohomology groups of a fibre space with fibre a space of type $\mathscr{K}(\pi, n)$. II. Trans. Amer. Math. Soc. **91** (1959), 505–524.

The author gives a proof, in the context of Serre cubical singular theory, of the following theorem of Hirsch: given a commutative ring with unit, K, a fibre-space $F\to E\to B$ with $\pi_1(B)$ operating trivially on $H^*(F;K)$, there exists a coboundary operator ∂ on $C^*(B;K)\otimes_K H^*(F;K)$ such that the resulting graded cohomology module is isomorphic to $H^*(E;K)$, provided $H^*(F;K)$ has a homogeneous K-basis, finite in each dimension [Bull. Soc. Math. Belgique **6** (1953), 79–96; MR **16**, 1142].

The theorem is applied to the case where F is an Eilenberg-MacLane space of type (π, n) and the coboundary operator is computed in the "stable range":

$$\partial(b\otimes 1)=\partial_B b\otimes 1,$$
$$\partial(b\otimes x)=\partial_B b\otimes x+(-1)^{p+1}\{b\cup T(y)k^{n+1}\}\otimes 1$$

where $b\in C^p(B;K)$, $x\in H^q(F;K)$ is a basis element, $p+q=m<2n$, y is the inverse image of x under suspension, $T(y)$ the cohomology operation corresponding to y, and k^{n+1} a cocycle representing the characteristic class of the fiber-space.

{Reviewer's note: The above computation is also an immediate consequence of more recent results of E. H. Brown [review below] and V. K. A. M. Gugenheim [Illinois J. Math., to appear].}

J.-P. Meyer (Baltimore, Md.)

4423:

Brown, Edgar H., Jr. Twisted tensor products. I. Ann. of Math. (2) **69** (1959), 223–246.

Let $p:X\to B$ denote a fibering with fiber F and let S denote the singular chains functor. The main objective of this paper is to show that $S(B)\otimes S(F)$ can be given a "twisted" differentiation to make it chain equivalent to $S(X)$ for a special class of fiberings which includes locally trivial fiber spaces. This special type of fibering is called a weakly transitive fiber space and since the definition employed is not quite strong enough to imply the results of the paper a few comments on this are in order.

Let B denote a topological space and $P(B)$ the Moore paths on B. We identify $b\in B$ with the Moore path $(\tilde{b}, 0)$, where $\tilde{b}(0)=b$, and α with (α, r). Furthermore, if α is a Moore path, we set $\varepsilon\alpha=\alpha(r)$. Now, let $p:X\to B$ denote a map and $U_p\subseteq P(B)\times X$ be given by $U_p=\{(\alpha, x)\,|\,\varepsilon\alpha=p(x)\}$. A "lifting function" for p is a map $\lambda:U_p\to X$ with $p\lambda(\alpha, x)=\alpha(0)$. Note that this definition of lifting function does not imply the universal covering homotopy theorem. A lifting function is called transitive (p. 225) if

$$\lambda(b, x)=x,\quad x\in X,\quad b=p(x), \tag{1.1}$$

$$\lambda(\alpha\beta, x)=\lambda(\alpha, \lambda(\beta, x)), \tag{1.2}$$

whenever $\alpha\beta$ is defined and $(\beta, x)\in U_p$. (With regard to this definition, it should be remarked that certain results of J. Schlesinger [Proc. Amer. Math. Soc. **9** (1958), 507–510; MR **20** #1316] preclude the existence of transitive lifting functions for locally trivial fibrations, in general. Schlesinger's results employ ordinary paths but it is not difficult to obtain similar results for Moore paths employing slightly different techniques.) Now, let $b_0\in B$ denote a fixed base point. According to the author, a lifting function "λ is weakly transitive if (1.1) holds for $b=b_0$ and (1.2) holds when $\varepsilon\alpha=\beta(0)=b_0$". This definition is too weak to validate the main theorems in the paper. The definition should require (1.1) for every $b\in B$ just as in the transitive case. Essentially, this is because a lifting function with property (1.1) implies the universal covering homotopy theorem. We will pinpoint the need for (1.1) further on. Now, $p:X\to B$ is called a weakly transitive fiber space if p admits a weakly transitive lifting function (in the revised sense). The following theorem, whose proof is substantially correct, is given. Theorem: If $p:X\to B$ is locally trivial and B is paracompact, then p admits a weakly transitive lifting function.

Let K be a DGA coalgebra and let A be a DGA algebra. A "twisting cochain" is a cochain $\varphi=\sum\varphi_q\in C^*(K, A)$ such that $\varphi_q\in C^q(K, A)$, $\varphi_0=0$, $\varphi_q(K_q)\subseteq A_{q-1}$, $a\varphi_1=0$ and

$$\partial\varphi_q=\varphi_{q-1}\partial-\sum_{k=1}^{q-1}(-1)^k\varphi_k\cup\varphi_{q-k},$$

where the cup product is formed by multiplication in A.

Suppose further that L is a DGA A-module, i.e., A acts nicely on L; then in the DGA module $K \otimes L$ a "twisted" differentiation ∂_φ is introduced as follows:

$$\partial_\varphi(k \otimes l) = \partial k \otimes l + (-1)^{\dim k}[k \otimes \partial l + (k \otimes l) \cap \varphi],$$

where the cap product depends upon the given pairing $A \otimes L \to L$. $K \otimes L$ together with ∂_φ is denoted by $K \otimes_\varphi L$ and is called the twisted tensor product of K and L.

The main result of the paper proceeds as follows. It is shown that if B is pathwise connected, then there is a twisting cochain φ for the DGA coalgebra $S(B)$ and the DGA algebra $S(\Omega(B))$, where $\Omega(B)$ is the space of Moore loops of B based at $b_0 \in B$. Now, let $p: X \to B$ denote a weakly transitive fiber space with fiber $F = p^{-1}(b_0)$ and let λ denote a weakly transitive lifting function for p. Then $S(F)$ becomes an $S(\Omega(B))$-module, since λ induces an action of $\Omega(B)$ on F.

Main theorem: There is a natural chain equivalence $\psi: S(B) \otimes_\varphi S(F) \to S(X)$.

The proof proceeds by the methods of acyclic models in the category of weakly transitive fiber spaces $\mathscr{F} = \{X, p, B, \lambda\}$. (1.1) is needed to show that the functor $\mathscr{F} \to S(X)$ is representable or alternately to define the chain homotopy inverse for ψ (which is left to the reader). It is interesting to note that the weakly transitive condition derived from (1.2) is crucial in proving $\partial_\varphi \partial_\varphi = 0$.

Several applications of the main theorem are given. For example, the author derives the Leray-Serre theorem on the spectral sequence of a fiber space [J.-P. Serre, same Ann. **54** (1951), 425–505; MR **13**, 574], its subsequent extension by Hurewicz and the reviewer [ibid. **68** (1958), 314–347; MR **21** #2237a], a few "less known" exact sequences, a theorem of Hirsch and a theorem concerning the bar and cobar constructions.

The paper contains several misprints.

E. Fadell (Madison, Wis.)

4424:

Shih, Weishu. Sur les complexes généralisés et la suite spectrale d'un fibré. C. R. Acad. Sci. Paris **248** (1959), 2935–2936.

The author defines a "generalised complex" to be an object E of an abelian category, provided with endomorphisms f_i satisfying suitable axioms; these axioms are so arranged that such a generalised complex determines a spectral sequence, in which the differential d_i is obtained from f_i by passing to the quotients. The author states that the spectral sequence of any fibering can be obtained in this way, by defining suitable endomorphisms f_i on the group $C(F) \otimes C(B)$. The precise construction is not given, but it appears from an exposé in the Lille Colloquium, 1959, that it is of some interest, since it includes the results of Hurewicz and Fadell [Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 241–245; MR **18**, 752] and may perhaps be related to results of E. H. Brown.

J. F. Adams (Cambridge, England)

4425:

Ganea, Tudor; and Hilton, Peter J. On the decomposition of spaces in Cartesian products and unions. Proc. Cambridge Philos. Soc. **55** (1959), 248–256.

In the Eckmann-Hilton duality theory [C. R. Acad. Sci. Paris **246** (1958), 2444–2447, 2555–2558, 2991–2993; MR **20** #6694–6696] the dual of a Cartesian product $Y_1 \times \cdots \times Y_n$ is the "union with base-points identified" $Y_1 \vee \cdots \vee Y_n$, and the dual of a "suspension" is a "loop-space". The authors prove, for connected non-contractible CW-complexes, the dual results: (A) $Y^n = Y \times \cdots \times Y$ ($n \geq 2$) never belongs to the homotopy type of a suspension; (A*) ${}^nY = Y \vee \cdots \vee Y$ ($n \geq 2$) never belongs to the homotopy type of a loop-space. Theorem (A) is deduced from a theorem about the Lusternik-Schnirelmann category: (B) cat $Y^n \geq n$. This suggests to the authors the invention of "cocategory", a concept dual to category. However they only succeed in proving the dual theorem (B*) cocat ${}^nY \geq n$ in the case $n = 2$.

Borsuk has conjectured that the homotopy type of a compact connected ANR does not have Cartesian decompositions into n factors for arbitrarily large n. The authors show that this is true if the fundamental group has elements of order p for only a finite number of primes p. They also prove the "dual" of Borsuk's conjecture: a compact connected ANR does not have a $\vee$-decomposition into n "summands" for arbitrarily large n. Examples are given to show that (A), (A*) and (B) fail if Y^n, ${}^nY$ are replaced by $Y_1 \times \cdots \times Y_n$, $Y_1 \vee \cdots \vee Y_n$. Also it is shown that cat Y^n is not determined by n and cat Y.

R. H. Fox (Princeton, N.J.)

4426:

Berstein, I.; and Ganea, T. Remark on spaces dominated by manifolds. Fund. Math. **47** (1959), 45–56.

Let $f: X \to Y$ be a continuous map. $g: Y \to X$ is a left homotopy inverse to f if gf is homotopic to the identity map of X. The authors' main theorem is the following: if f has a left homotopy inverse, Y is a compact n-dimensional manifold, and if $H^n(X) \neq 0$, then f is a homotopy equivalence. The main lemma used in proving this theorem is the following proposition. Let P and Q be regular covering spaces of a connected CW-complex X and of a compact n-dimensional manifold Y respectively. Let $\Pi(P, p)$ be the group of covering translations of $p: P \to X$. Let $e: P \to Q$ and $f: X \to Y$ be such that $qe = fp$. Let G be either Z or Z_2 and assume further that ${}_1e: \Pi(P, p) \to \Pi(Q, q)$ is an isomorphism, $H_n(Y; G) \neq 0$, and $f_*: H_n(X; G) \to H_n(Y; G)$ is onto. Then $e_*: H_r(P; G) \to H_r(Q; G)$ is onto for all $r \geq 0$.

F. P. Peterson (Cambridge, Mass.)

4427:

Ganea, T. On ε-maps onto manifolds. Fund. Math. **47** (1959), 35–44.

Let $f: X \to Y$ be a continuous map, where X is a compact metric space. f is an ε-map if diam $(f^{-1}(y)) < \varepsilon$ for all $y \in Y$. The author proves several theorems showing that the existence of ε-maps of X onto closed n-manifolds for all $\varepsilon > 0$ is a strong condition. For example, suppose X is an n-dimensional ANR, and suppose for every $\varepsilon > 0$ there is an ε-map onto a closed n-manifold (depending on ε). Then (1) X is of the same homotopy type as a closed n-manifold, (2) if A is a closed proper subset of X, then $H^n(A) = 0$, and (3) there exists $\sigma > 0$ such that every closed subset A of X with diam $A < \sigma$ and $H^{n-1}(A; Z_2) \neq 0$ separates X. This answers negatively a question of Ulam asking whether there exists for every $\varepsilon > 0$ an ε-map of the disc onto a 2-dimensional torus.

F. P. Peterson (Cambridge, Mass.)

4428:

Weier, Joseph. Eine Homologieeigenschaft stetiger

Abbildungen von Mannigfaltigkeiten. Rev. Mat. Hisp.-Amer. (4) **18** (1958), 157–170.

Let P and Q be triangulable orientable compact manifolds, dim $P=n+1$, dim $Q=n$. Given any map $f: P \to Q$, two arbitrarily close approximations f_1, f_2 of f are constructed in such a fashion that the set $\{x \mid x \in P, f_1(x)=f_2(x)\}$ is a 1-polyhedron A having no isolated points [this completes the proof of a proposition treated by the author in Rend. Circ. Mat. Palermo (2) **6** (1957), 343–348; MR **20** #4271; and announced in C. R. Acad. Sci. Paris **239** (1954), 1111–1113; MR **16**, 503]. The author also considers an integral chain $\sum \xi_i x_i$ associated with (f_1, f_2), where x_i are oriented 1-simplexes of A and ξ_i integers determined as a sort of indexes belonging to the coincidences of f_1 and f_2. It is proved that $\sum \xi_i x_i$ is a cycle. {Reviewer's remark: For this the author says that "the chain $\sum \xi_i x_i$ is zero homologous". The author's notion of "algebraic graph" coincides with that of a cycle, which makes the use of "spherical coverings" superfluous.}

S. Mardešić (Zagreb)

4429:

★**Thom, René. La classification des immersions.** Séminaire Bourbaki; 10e année: 1957/1958. Textes des conférences; Exposés 152 à 168; 2e éd. corrigée, Exposé 157, 11 pp. Secrétariat mathématique, Paris, 1958. 189 pp. (mimeographed)

The reviewer [Ann. of Math. (2) **69** (1959), 327–344; MR **21** #3862] has proved that under regular homotopy equivalence, the set of immersions of S^k in E^n is in an isomorphic correspondence with $\pi_k(V_{n,k})$ (the kth homotopy group of the Stiefel manifold of k-frames in E^n). His proof is broken into two parts, "the covering homotopy theorem" and "the weak homotopy equivalence theorem". The author gives an exposition of this work but essentially is concerned with the covering homotopy theorem, which is the more difficult of the two parts of the reviewer's proof. The author does more than give an exposition but clarifies the proof by isolating the difficult part and by breaking down the proof into several steps. The result is a more conceptual proof of the covering homotopy theorem. There are some gaps which can be filled in but which make for some difficulty in reading the author's work.

S. Smale (Rio de Janeiro)

4430:

Bott, Raoul; and Samelson, Hans. Applications of the theory of Morse to symmetric spaces. Amer. J. Math. **80** (1958), 964–1029.

Results announced by the authors [Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 490–493; MR **17**, 182] are proved in the present paper. Let Ω be the loop space on the connected compact Lie group K and let S be the set of geodesics on K leading from a general point to the identity. Then an explicit isomorphism γ of S_*, the free module generated by the elements of S, into $H_*(\Omega; Z)$, the integral singular homology group of Ω, is given. To carry out the program, the authors deal with the loop space of a manifold M on which a compact Lie group K acts with at least one fixed point. Then it is proved that if the action of K on M satisfies a certain infinitesimal condition, called variational completeness, there is an isomorphism γ of S_* onto $H_*(\Omega; Z)$ provided that S is K-orientable (or, without assuming the K-orientability of S, an isomorphism of $\bar{S}_*$, the vector space with integer mod 2 coefficients generated by the elements of S, onto $H_*(\Omega; Z_2)$, the singular homology group of Ω with integer mod 2 coefficients). The validity of variational completeness is established in various instances all of which are associated with symmetric spaces. The rest of the paper is devoted to applications of these results to the topology of Lie groups and symmetric spaces in general.

{Editors' note: The authors wish to point out the following misprints. (1) Theorem IV (b) (p. 995): add "B_2" after "B_1". (2) P. 1019, l. −10: $\mathfrak{k}$ instead of k; l. −7: $\mathfrak{k}$ instead of t. (3) P. 1024, l. 1: $\mathfrak{g}_{\mathfrak{a}^2}{}^{\perp}$ instead of $\mathfrak{g}_{\mathfrak{a}^2}$.}

C. T. Yang (Philadelphia, Pa.)

DIFFERENTIAL GEOMETRY, MANIFOLDS

See also 4430, 4874, 4875.

4431:

★**Sándor, I. Über die natürliche Gleichung der Raumkurve in der elliptischen Geometrie.** Einige Arbeiten des Lehrstuhles für Mathematik im Lehrjahre 1956/57, pp. 25–28. Wissenschaftliche Veröffentlichungen der Technischen Universität für Bau- und Verkehrswesen in Budapest, Budapest, 1958. 80 pp.

Zwei Raumkurven, bei denen in Punkten gleicher Bogenlänge sowohl Krümmung als Torsion übereinstimmen, sind deckungsgleich.—Raumkurven, die in entsprechenden Punkten gleiche Hauptnormale besitzen, werden Bertrandsche Adjungierte genannt. Entsprechende Punkte solcher Kurven haben konstante Entfernung; in entsprechenden Punkten gezogene Tangenten schliessen einen konstanten Winkel ein. Bei einer Bertrandschen Kurve besteht zwischen Krümmung und Torsion eine lineare Beziehung.

F. Bachmann (Kiel)

4432:

Mayer, O. Sur les surfaces réglées de Peterson. Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) **2 (50)** (1958), 61–74.

L'Autore si occupa di superficie di Peterson (superficie P), e cioè superficie che ammettono una rete coniugata P_2 di linee coniche, cioè due sistemi coniugati tali che le sviluppabili circoscritte alla superficie lungo quelle linee siano coni. Precisamente l'Autore determina le superficie P analitiche che siano rigate e che soddisfino ad opportune ipotesi di genericità (siano ricoperte semplicemente da reti P_2 analitiche ed abbiano su ogni generatrice solo punti singolari isolati).

Premesso che per una superficie siffatta i luoghi dei vertici dei coni circoscritti lungo la rete P_2 sono curve razionali d'ordine $n \leq 4$, si ottiene—attraverso un'accurata analisi—che le rigate P in questione sono quelle rappresentabili con una delle equazioni seguenti: $z=x^\alpha y$ $(\alpha \neq 0)$; $x=ye^z$; $\log(z/y)+1/y-x/z=0$; $y=x \operatorname{tg} z$; $y=x \operatorname{tgh} z$ ed inoltre le rigate cubiche di Cayley.

D. Gallarati (Genoa)

4433:

Franckx, Ed. Sur les surfaces réglées gauches. Bull. Soc. Roy. Sci. Liège **28** (1959), 138–140.

Tale nota fa seguito ad un' altra [lo stesso Bull. **24** (1955),

296–300; MR 17, 780] in cui era stata dimostrata una certa proprietà metrica dei punti della linea di stringimento di una superficie rigata. Qui si inverte il risultato mostrando che quella proprietà è caratteristica dei punti centrali delle generatrici della rigata.

F. Gherardelli (Florence)

4434:

Maxfield, John E.; and Selfridge, R. G. The surface area of a screw. Appl. Sci. Res. A 8 (1959), 377–385.

"Formulae are derived for surface area of a right cylindrical screw having as cross-section edge straight lines and/or circular segments. The formulae involve non-elementary functions of the incomplete integral of the first, second and third kinds. These integrals are tabulated in the literature." (From the authors' abstract)

M. D. Friedman (Needham Heights, Mass.)

4435:

Zalts, K. Ya. Plane transformations preserving areas. Latvijas Valsts Univ. Zinātn. Raksti 8 (1956), no. 2, 5–10. (Russian. Latvian summary)

The author remarks that the condition for area-preserving transformations in cartesian coordinates $x(u, v)$, $y(u, v)$ is that the absolute value of the Jacobian is unity. He gives several geometrical constructions for this kind of transformation, transforming the point $P(u, v)$ into Q, which are: (1) drawing from P the tangent PT to a given curve and taking $PT=TQ$; (2) O being a fixed point, taking the point Q on the circle about O with radius OP at a constant distance measured along the arc; (3) drawing the perpendicular from P to the u-axis and taking the point Q at the distance $|v|$ on the circle about the origin through the foot of the perpendicular measured along the arc from the foot; (4) taking $PQ=a$ on the perpendicular to OP at P.

E. M. Bruins (Amsterdam)

4436:

Osserman, Robert. Proof of a conjecture of Nirenberg. Comm. Pure Appl. Math. **12** (1959), 229–232.

In 1927 S. Bernstein proved that a minimal surface which can be represented in the form $z=f(x, y)$ for all x and y must be a plane [Math. Z. **26** (1927), 551–558]. In the sequel a number of different proofs for this famous theorem have been given. Obviously, the surface $z=f(x, y)$ is simply connected, complete in its metric, and its spherical image is contained in a semi-sphere. In this paper the author proves—thus answering in the affirmative a conjecture by L. Nirenberg—the more general theorem: "A simply connected complete minimal surface in 3-space whose spherical image omits a whole neighborhood of a point must be (of parabolic type and hence) a plane." First the following lemma is ascertained: If S is a non-compact simply connected complete surface with metric ds^2 and if there exists a real-valued function $f(p)$ on S satisfying $\Delta_B \log f=K$, $f\geqq\varepsilon>0$, then S is conformally equivalent to the plane. (Δ_B stands for the second Beltrami operator, and K is the Gaussian curvature.) Let now S be a minimal surface with the properties mentioned above, its spherical image omitting a neighborhood of the north pole of the unit sphere Σ. Map Σ stereographically onto the z-plane and define on S the function $f(p)=2/(1+|z|^2)$ where $z=z(p)$ is the image point of p under the composed map $S\to\Sigma\to z$ of S into the z-plane. Then $\Delta_B \log f=K$ and $f\geqq\varepsilon>0$. By virtue of the lemma, S can be mapped conformally and in a one-to-one manner onto the whole w-plane. So an analytic mapping of the w-plane onto a bounded region of the z-plane is obtained. Its mapping function must be a constant, so that S has a single normal direction and must be a plane.

J. C. C. Nitsche (Minneapolis, Minn.)

4437:

Tevzadze, G. P. A problem of the theory of nets and some classes of surfaces in a projective space. Soobšč. Akad. Nauk Gruzin. SSR **22** (1959), 9–15. (Russian)

On trouve les réseaux conjugués d'une surface de S_3 à vecteur de Tchebisheff donné; pour les surfaces possédantes une infinité de réseaux dont les axes de Green coïncident on a $\beta_\gamma=\frac{1}{4}(u+v)^{-2}$. On trouve β, γ des surfaces de A_3 qui possèdent ∞^1 de réseaux conjugués de Tchebisheff et on démontre le théorème de Finikoff: Seulement les hélicoïdes minimaux possèdent une infinité de réseaux de Voss. L'A. utilise la méthode de M. Norden (des surfaces normalisées).

A. Švec (Prague)

4438:

Čech, Eduard. Déformation projective des congruences W. Czechoslovak Math. J. **6 (81)** (1956), 401–414. (Russian summary)

Dualizzazione di una data congruenza L di rette di un S_3 proiettivo è la trasformazione sviluppabile che si ottiene sostituendo ad ogni retta g di L la corrispondente duale g^* in una correlazione dello S_3. L'A. nella Memoria, Transformations développables des congruences des droites [lo stesso J. **6** (81) (1956), 260–286; MR **19**, 676], introdotto per esse la nozione di elemento lineare proiettivo, ha caratterizzato le dette trasformazioni come tutte e sole trasformazioni che conservano l'elemento lineare proiettivo.

Ciò permette all'A. di dimostrare, nella presente Memoria, che la dualizzazione di una congruenza L è una trasformazione proiettiva se e solo se L è una congruenza W, ossia se e solo se L determina una corrispondenza asintotica tra le due superficie focali.

L'A. caratterizza quindi le congruenze W appartenenti ad un complesso lineare, e quelle per le quali la dualizzazione è semisingolare, dimostrando anche che le congruenze W a dualizzazione semisingolare dipendono da quattro funzioni arbitrarie di una variabile. Inizia poi lo studio della determinazione di tutte le congruenze W con dato elemento lineare proiettivo, assegnando di questo non facile problema solo risultati parziali, ma importanti ed interessanti. Tra questi risultati, in particolare ricordo: (1) ciascuna congruenza W ammette al più ∞^6 deformate proiettive le quali non appartengono ad un complesso lineare e non sono a dualizzazione asintotica; (2) ciascuna congruenza W a dualizzazione asintotica ammette deformate proiettive che sono a dualizzazione asintotica dello stesso tipo e che dipendono da una funzione arbitraria di una variabile; in particolare tra le dette deformate proiettive ve n'è una appartenente ad un complesso lineare.

C. Longo (Parma)

4439:

Stephanidis, N. K. Über ein Vektorfeld eines Strahlensystems. Arch. Math. **10** (1959), 221–225.

The rays of a congruence may be given by the vector $\mathbf{a}(u, v)$ to one of its points and the unit vector $\mathbf{a}'(u, v)$ in its direction. Plotting the unit vector from a common origin there is a one to one correspondence between the rays and part of the unit sphere. The author shows that the divergence of $\mathbf{a}'$ is a geometrical invariant equal to twice the mean curvature of the ray system.

M. Herzberger (Rochester, N.Y.)

4440:

Mihăilescu, Tiberiu. Géométrie différentielle affine des courbes planes. Czechoslovak Math. J. **9** (**84**) (1959), 265–288. (Russian summary)

L'auteur utilise la méthode du repère mobile sous la forme donnée par E. Cartan [*La théorie des groupes finis et continus*, Gauthier-Villars, Paris, 1937] pour présenter une étude systématique des propriétés différentielles des courbes planes, invariantes par rapport au groupe affine général du plan. Il associe à chaque point de la courbe des classes de repères de plus en plus spécialisés, les repères d'une classe p étant équivalents par rapport à un sous-groupe affine à p paramètres; il retrouve ainsi des notions déjà introduites par des auteurs comme E. Cartan et W. Blaschke et en précise la signification géométrique: normale affine, arc affine, invariant de courbure affine. Il définit des notions nouvelles: le centre affine d'une courbe de courbure affine constante, le rapport de trois points d'une courbe qui résulte de la considération des développantes de Cartan, et les courbes affinement parallèles à une courbe donnée. *M. Decuyper* (Lille)

4441:

Nožička, Frant. Sur le contact des hypersurfaces dans un espace affine linéaire. Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) **1** (**49**) (1957), 337–353.

"Dans cet article on introduit une certaine notion de contact de deux hypersurfaces plongées dans un espace affine linéaire. La notion de contact qu'on veut définir, est invariante dans le sens de la géométrie affine. On trouve certaines conditions nécessaires et suffisantes très simples pour que deux hypersurfaces considérées aient un contact d'ordre k au moins." *Résumé de l'auteur*

4442:

Mammana, Carmelo. L'insieme delle varietà immagini delle omografie di S_n dei vari tipi, come insieme parzialmente ordinato. Matematiche. Catania **12** (1957), 35–50.

In un precedente lavoro [Ann. Scuola Norm. Sup. Pisa (3) **11** (1957), 249–263; MR **20** #4230] l'Autore ha determinato i tipi di omografie di S_n di cui una data omografia si può riguardare come limite, ottenendo risultati che trovano applicazione nella nota presente. Interpretate le omografie di S_n come punti di uno spazio proiettivo, l'insieme delle varietà W_γ delle omografie di data caratteristica γ si può rendere parzialmente ordinato per inclusione; e risulta che $W_{\gamma'} \subseteq W_{\gamma''}$ se e solo se un'omografia di caratteristica γ'' è limite di omografie di caratteristica γ'. Determinate le condizioni necessarie e sufficienti affinchè una varietà W_γ copra una W_φ, l'A. mostra che gli insiemi parzialmente ordinati delle varietà W_γ sono subordinati in un reticolo se $n \leq 3$ (distributivo se $n = 1, 2$; non distributivo nè modulare se $n = 3$), cosa che non è più vera per $n \geq 4$. Le varietà W_γ delle omografie di S_n con la caratteristica formata da un sol gruppo caratteristico costituiscono un sottoinsieme subordinato in un reticolo distributivo se $n \leq 5$, non distributivo nè modulare se $n \geq 6$. *D. Gallarati* (Genoa)

4443:

Švec, Alois. Courbes de Darboux d'une hypersurface. Časopis Pěst. Mat. **84** (1959), 162–164. (Czech. Russian and French summaries)

Let S_{n+1} and S^*_{n+1} be dual projective spaces, $\pi \subset S_{n+1}$, $\pi^* \subset S^*_{n+1}$ dual hypersurfaces, H a collineation between S_{n+1} and S^*_{n+1} tangent to the mapping $\pi \to \pi^*$ and such that the H-linearizing straight lines are tangent to π^*. There is a set of ∞^{2n+1} collineations H of the correspondence $\pi \to \pi^*$ for which the H-linearizing straight lines of the tangents to π are tangents to π^*. The tangents to π, coincident with their H-linearizing straight lines, constitute at every point a cubic cone for each H. There is a set of ∞^{n+1} H's for which this cone is apolar to the asymptotic cone. Then the cubic cone is the cone of Darboux' directions. [For the concepts used in this review, see G. F. Laptev, Trudy Moskov. Mat. Obšč. **2** (1953), 275–382; MR **15**, 254; and E. Čech, Časopis Pěst. Mat. Fys. **74** (1949), 32–48; MR **12**, 534].

V. Hlavatý (Bloomington, Ind.)

4444:

Hu, Hou-sung. A new geometry of a space of K-spreads. Sci. Record (N.S.) **3** (1959), 107–111.

A new concept of K-spread is introduced; it is called projective, and is an intermediate between the descriptive and affine geometries. It is shown that a fractional linear transformation of the parameters of the K-spread is equivalent to the change in Γ's which is usually called projective (torsion- and path-preserving). A space of K-spreads is projectively flat if the equations can be transformed to a linear form, with suitable coordinates and parameters. A necessary and sufficient condition is $Z_{jk}{}^{t}|_{\alpha^\beta} = 0$, where $Z_{jk}{}^{t}$ are the fundamental projective invariants. *A. Nijenhuis* (Seattle, Wash.)

4445:

Klingenberg, W. Contributions to Riemannian geometry in the large. Ann. of Math. (2) **69** (1959), 654–666.

Rauch [same Ann. **54** (1951), 38–55; MR **13**, 159] proved that if the sectional curvatures of a complete simply-connected Riemannian manifold M lie between bounds of ratio h, where h is the root of $\sin(\pi\sqrt{h}) = (\sqrt{h})/2$, then M is homeomorphic to a sphere. The author sharpens the constant, in the even-dimensional case, to the solution of $\sin(\pi\sqrt{h}) = \sqrt{h}$. The proof depends on the following 'converse' of the Bonnet-Myers theorem, interesting in its own right: If M is a simply-connected compact Riemannian manifold with everywhere positive sectional curvature bounded by L, then the diameter of M is at least $\pi/\sqrt{L}$. This is proved following the outline given by Mutō [Sci. Rep. Yokohama Nat. Univ. Sect. I **2** (1953), 1–12; MR **15**, 827]. The proofs are accomplished by means of detailed examinations of the cut locus. {In a later paper [#4446] the author improves the Rauch constant for odd-dimensional manifolds, too. However, Mutō's theorem for odd dimensions is still open.} *L. W. Green* (Minneapolis, Minn.)

4446:

Klingenberg, Wilhelm. Über kompakte Riemannsche Mannigfaltigkeiten. Math. Ann. **137** (1959), 351–361.

The major result in this paper is the extension of the author's previous generalization of the Rauch theorem for spheres [Proc. Nat. Acad. Sci. U.S.A. **44** (1958), 586–588; MR **20** #2753] from the case of even dimensions to the case of all dimensions.

L. Auslander (Bloomington, Ind.)

4447:

Tonolo, Angelo. Classi di ds^2 ternari le cui geodetiche ammettono integrali primi quadratici. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **24** (1958), 230–238.

Il problema di determinare il tensore fondamentale $a_{\lambda\mu}(x)$ di una varietà riemanniana V_n le cui geodetiche ammettono un integrale primo quadratico $c_{\lambda\mu}\dot{x}^\lambda\dot{x}^\mu = \text{cost}$ è stato tradotto da Levi-Civita (nel 1900) in un sistema di equazioni differenziali a derivate parziali del primo ordine involgente i seguenti elementi determinati dal tensore doppio simmetrico covariante $c_{\lambda\mu}$: (1) gli autovalori ρ_h; (2) il campo covariante de autovettori unità mutuamente ortogonali; (3) i coefficienti di rotazione della terna ortogonale di linee individuata dagli autovettori.

Nella presente Nota l'A. integra, nel caso $n=3$, il detto sistema di equazioni differenziali, determinando i relativi ds^2 della V_3 riemanniana in ciascuna delle seguenti ipotesi: (1) autovalori ρ_h distinti e terna di linee ortogonali costituita da congruenze normali (ds^2 di Stäckel); (2) due delle ρ_h uguali, ma distinte dalla terza, e congruenza delle linee oltre che normali anche isotrope (ds^2 di Painlevé); (3) due delle ρ_h uguali, ma distinte dalla terza, ed una delle congruenze di linee normale e geodetica (ds^2 di Levi-Civita).

C. Longo (Parma)

4448:

Sen, R. N. A note on symmetric affine connections. Indian J. Math. **1**, no. 1, 17–19 (1958).

Let g_{ij} be the metric tensor of a Riemann space and Γ^h_{ij} arbitrary coefficients of connection. Define $\bar{\Gamma}^h_{ij}$ by $\bar{\Gamma}^h_{ij} = \Gamma^h_{ij} + g^{hk}g_{ik,j}$, the comma denoting differentiation with respect to the Γ^h_{ij}. The author considers the consequences of the assumption that both Γ^h_{ij} and $\bar{\Gamma}^h_{ij}$ are symmetric in i, j. Results are stated in terms of certain orthogonal ennuples.

A. Fialkow (Brooklyn, N.Y.)

4449:

Golab, S. Über die Metrisierbarkeit der affin-zusammenhängenden Räume. Tensor (N.S.) **9** (1959), 1–7.

The author considers the problem of determining when a connected affine space A_n is a metric space (a Riemannian space, V_n). As noted by the author, this problem was first studied by Eisenhart and Veblen [Proc. Nat. Acad. Sci. U.S.A. **8** (1922), 19–23] who gave necessary and sufficient conditions when the rank, p, of the metric tensor is n. In this paper, the author considers the case when q, the maximum of p, is less than or equal to n, for all tensors, $g_{\alpha\beta}$, associated with a given connection, $\Gamma_{\lambda\mu}{}^\rho$, by the partial differential equations

$$\partial_\mu g_{\lambda\nu} - g_{\rho\nu}\Gamma_{\lambda\mu}{}^\rho - g_{\lambda\rho}\Gamma_{\nu\mu}{}^\rho = 0.$$

Further, the author limits the present analysis to the integrability conditions when $n=2$, $q=0$, 1, 2. This case is completely discussed and results are shown to depend on a matrix of rank r and on two densities. The $g_{\alpha\beta}$ are determined.

N. Coburn (Ann Arbor, Mich.)

4450:

Lichnerowicz, André. Transformations affines et holonomie. C. R. Acad. Sci. Paris **244** (1957), 1868–1870.

Let V_m be a differentiable manifold, let $x \in V_m$ and let T_x be the tangent space at x. In case V_m has a Riemannian structure the reviewer has defined [Trans. Amer. Math. Soc. **80** (1955), 528–542; Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 258–261, 354–357; MR **18**, 930; **19**, 454] a Lie algebra of endomorphisms of T_x using the Killing fields on V_m and has shown that this algebra is identical with the holonomy algebra whenever V_m is compact homogeneous. In this note the author indicates what the situation is under more general circumstances. Let ω be an affine connection on V_m and let $\bar{\omega}$ be the associated affine connection with the opposite torsion. Then if X is any (ω) infinitesimal affine transformation (i.a.t.) let $A_X(x)v = -\bar{\Delta}_v X$ where $v \in T_x$ and $\bar{\Delta}_v$ is $\bar{\omega}$-covariant differentiation with respect to v. Now if $V_m = G/H$ and ω is G-invariant, let $K_x(G, \omega)$ be the group of endomorphisms of T_x whose Lie algebra is the one generated by all $A_X(x)$ when X runs through all i.a.t. corresponding to the Lie algebra of G. It is asserted that $K_x(G, \omega)$ lies between the restricted homogeneous holonomy group σ_x at x and its normalizer, and that the normalizer of $K_x(G, \omega)$ contains the full homogeneous holonomy group Ψ_x as well as the linear isotropy group H_x at x.

It is also stated that a tensor field t on G/H which satisfies any two of the following conditions also satisfies the third: (a) t is G-invariant, (b) t is covariant constant, (c) $t(x)$ is invariant under $K_x(G, \omega)$ for some point x. An application is given. Further relations between Ψ_x, H_x and $K_x(G, \omega)$ are stated when further assumptions are made. Finally, the reviewer's theorem that $\Psi_x = H_x K_x(G, \omega)$, in the compact Riemannian case, although not generalized, is put in the context of these relations.

B. Kostant (Berkeley, Calif.)

4451:

Lichnerowicz, André. Transformations analytiques d'une variété kählérienne et holonomie. C. R. Acad. Sci. Paris **245** (1957), 953–956.

Soit V une variété kählérienne compacte. Désignons par d' et δ' les opérateurs de différentiation et de codifférentiation de type $(1, 0)$ et $(-1, 0)$ respectivement, par Δ le laplacien, par Q le double de l'opérateur associé au tenseur de Ricci et par $d\tau$ l'élément de volume de V. Théorème 1: Si le premier nombre de Betti de V est nul ou si la caractéristique d'Euler-Poincaré de V est non nulle, il existe un isomorphisme de l'algèbre du plus grand groupe connexe de transformations analytiques de V sur une algèbre de Lie définie sur l'espace des solutions ρ à valeurs complexes de l'équation $\Delta\Delta\rho = 2\delta'Qd'\rho$ telles que $\int_V \rho d\tau = 0$. L'opérateur "presque complexe" agissant sur les vecteurs tangents en $x \in V$ définit un tenseur $F_i{}^j$; soit J l'opérateur réel défini sur les tenseurs covariants d'ordre 2 par $J: t_{ij} \to t_{ir}F^r_j$. Désignons par σ_x l'algèbre d'holonomie en $x \in V$; chaque élément de σ_x peut être identifié à une forme réelle de type $(1, 1)$; soit D_i l'opérateur de dérivation covariante de la connexion kählérienne de V. Théorème 2: Pour toute

1-forme ξ définissant une transformation infinitésimale analytique sur V, le tenseur réel $D_i\xi_j$ de type (1, 1) appartient, en chaque point $x \in V$, à l'espace vectoriel $\sigma_x + J\sigma_x$. Corollaire: Le plus grand groupe connexe de transformations analytiques de V respecte le multifeuilletage canonique de V défini par la réductibilité. Les démonstrations, esquissées ici, sont publiées intégralement au chapitre V, II du livre de l'auteur, *Géométrie des groupes de transformations* [Dunod, Paris, 1958].

P. Dolbeault (Bordeaux)

4452:

Lichnerowicz, André. Transformations analytiques et isométries d'une variété kählérienne compacte. C. R. Acad. Sci. Paris **247** (1958), 855–857.

Cette note fait suite à deux autres [A. Lichnerowicz, mêmes C. R. **244** (1957), 3011–3013; MR **20** #996; et note analysée ci-dessus] dont elle utilise les méthodes et les notations. Soit L l'algèbre des transformations infinitésimales (en abrégé t.i.) analytiques de la variété kählérienne compacte V et soit ξ une 1-forme définissant une t.i. analytique dont la partie de type (1, 0) est $\xi_{(1,0)}$. On désigne par M l'opérateur sur les 1-formes défini par la structure complexe. Théorème 1: À toute t.i. analytique ξ, on peut associer, d'une manière unique, un scalaire complexe χ vérifiant $\int_V \chi(\star 1) = 0$ et tel que $\xi_{(1,0)} = d'\chi + h$, où h est holomorphe fermée; a $M\xi$ correspond $i\chi$; pour que ξ définisse une isométrie infinitésimale, il faut et il suffit que χ soit imaginaire pur. Théorème 2: Soit I le sous-espace de L formé des t.i. pour lesquelles les $\xi_{(1,0)}$ sont d'-homologues à 0; l'idéal dérivé de L est contenu dans I qui est donc tel que L/I soit abélienne. Théorème 3: Soit L_1 l'algèbre des isométries infinitésimales de V; la sous-algèbre complexe $L_1 + ML_1$ est réductive dans L.

P. Dolbeault (Bordeaux)

4453:

Suguri, T.; Nakayama, S.; and Ueno, S. Some notes on unitary connections defined in Hermitian manifolds. Tensor (N.S.) **8** (1958), 184–195.

An Hermitian manifold $M^{(n)}$ of n complex dimensions is studied. § 1. For the unitary connection in $M^{(n)}$, a positive definite quadratic differential form $ds^2 = g_{j\lambda}dz^j dz^\lambda$ together with the Hermitian metric $ds^2 = 2g_{\alpha\beta^*}dz^\alpha dz^{\beta^*}$, the metric unitary connection Γ^i_{jk}, torsion tensor $S^{\cdot\cdot i}_{jk}$, the curvature tensors R^i_{jkl}, $R_{h\bar{j}k\bar{l}} = g_{hi}R^i_{jkl}$, the Bianchi identities and an additional new allied identity are explained and then a conformal relation of an Hermitian manifold $M^{(n)}$ to a Kählerian manifold $\tilde{M}^{(n)}$ is defined, indicating the resulting special forms of $R^\alpha_{\beta\gamma\delta}$, $S^{\cdot\cdot\alpha}_{\beta\gamma}$ and the curvature tensor. § 2. The sectional curvature is defined and the isotropic Hermitian manifold is studied, leading to the following three theorems. 1. A necessary and sufficient condition for the curvature tensor $R_{\alpha\beta^*\gamma\delta^*}$ of an isotropic Hermitian manifold $M^{(n)}$ $(n > 1)$ to vanish identically is that the torsion tensor is complex analytic. If the condition is satisfied, the isotropic sectional curvature K is also identically zero. 2. If an isotropic Hermitian manifold $M^{(n)}$ $(n > 1)$ is conformal to an isotropic Kählerian manifold $\tilde{M}^{(n)}$, then the curvature tensor $R_{\alpha\beta^*\gamma\delta^*}$ of $M^{(n)}$ vanishes identically. 3. If a Kählerian manifold $\tilde{M}^{(n)}$ $(n > 1)$ is conformal to an isotropic Hermitian manifold $M^{(n)}$ whose curvature tensor $R_{\alpha\beta^*\gamma\delta^*}$ vanishes identically, then the manifold $\tilde{M}^{(n)}$ is also isotropic. § 3. The holomorphic sectional curvature is defined and the holomorphically isotropic Hermitian manifold is studied, leading to the following three theorems: 4. An holomorphically isotropic Hermitian manifold $M^{(n)}$, whose torsion tensor is complex analytic and whose holomorphic sectional curvature is $K(z, \bar{z}) \neq 0$, is conformal to an holomorphically isotropic Kählerian manifold $\tilde{M}^{(n)}$, whose holomorphic sectional curvature ε is $+1$ or -1 according as $K(z, \bar{z}) > 0$ or $K(z, \bar{z}) < 0$. 5. Let an Hermitian manifold $M^{(n)}$ (with fundamental metric tensor $g_{\alpha\beta^*} = e^{-k}G_{\alpha\beta^*}$), be conformal to an holomorphically isotropic Kählerian manifold $\tilde{M}^{(n)}$ whose fundamental tensor is $G_{\alpha\beta^*}$ and whose holomorphic sectional curvature is ε. If $M^{(n)}$ is also holomorphically isotropic and its holomorphic sectional curvature is $K(z, \bar{z}) = \varepsilon e^k$, then the torsion of $M^{(n)}$ is complex analytic. 6. Let $G_{\alpha\beta^*} = \partial^2\Phi/\partial z^\alpha \partial z^{\beta^*}$ be the fundamental metric tensor of an holomorphically isotropic Kählerian manifold $\tilde{M}^{(n)}$ with the holomorphical sectional curvature $\tilde{K}$, and let $g_{\alpha\beta^*} = e^{-k}G_{\alpha\beta^*}$ be a fundamental metric tensor of an Hermitian manifold $M^{(n)}$ which is conformal to $\tilde{M}^{(n)}$. In order that $M^{(n)}$ be also holomorphically isotropic, it is necessary and sufficient that the real-valued function $k(z, \bar{z})$ is of the form $k(z, \bar{z}) = c\Phi(z, \bar{z}) + k_0(z, \bar{z})$, where c is an arbitrary real constant and $k_0(z, \bar{z})$ is an arbitrary real-valued function satisfying the condition $\partial^2 k_0/\partial z^\alpha \partial z^{\beta^*} = 0$. The holomorphic sectional curvature K of $M^{(n)}$ is given by $K = e^k(c + \tilde{K})$. § 4. Three types of Ricci tensors

$$E_{\alpha\beta^*} = R_{\alpha\beta^*} + \frac{\partial^2 k}{\partial z^\alpha \partial z^{\beta^*}}, \quad S_{\alpha\beta^*} = R_{\alpha\beta^*} + g_{\alpha\beta^*}\left(g^{\rho\sigma^*}\frac{\partial^2 k}{\partial z^\rho \partial z^{\sigma^*}}\right),$$

$$T_{\alpha\beta^*} = R_{\alpha\beta^*} + n\frac{\partial^2 k}{\partial z^\alpha \partial z^{\beta^*}}$$

($R_{\alpha\beta^*}$ for $\tilde{M}^{(n)}$) lead to three theorems concerning necessary and sufficient conditions for their coincidence. § 5. A special Hermitian manifold called the pseudo-Einstein manifold by S. I. Goldberg [Ann. of Math. (2) **64** (1956), 364–373; MR **18**, 598] is studied, leading to a theorem 9 concerning the conditions on an holomorphically isotropic Hermitian manifold $M^{(n)}$ for a conformally related $\tilde{M}^{(n)}$ to be an Einstein manifold and to two others: 10. If an Hermitian manifold $M^{(n)}$ is conformal to an Einstein manifold in the Kählerian sense, then $M^{(n)}$ is a pseudo-Einstein manifold of the type S (in the author's sense). 11. An Hermitian manifold which is suitably conformal to an Einstein manifold in the Kählerian sense can be made a pseudo-Einstein manifold of type E (of type T) (in the author's sense).

T. Takasu (Yokohama)

4454:

Atiyah, M. F. Some examples of complex manifolds. Bonn. Math. Schr. no. 6 (1958), 28 pp.

"The object of this talk is to construct and study examples of compact differentiable manifolds which can carry both kählerian and non-kählerian complex structures. These examples are due to A. Blanchard and independently to E. Calabi. The present treatment was developed by R. Bott and the author.

"The basic idea is very simple. We consider the set J_n of all complex vector space structures on R^{2n}. It is well known, and can be proved in many ways, that J_n is endowed with a natural structure of complex manifold. Each $J \in J_n$ defines a complex structure T_J on the real torus R^{2n}/Z^{2n}. What has then to be shown is that the union of all the T_J for $J \in J_n$ is again a complex manifold M_n

with a natural projection onto J_n. Although J_n is not compact it contains compact complex (algebraic) submanifolds of positive dimension if $n \geq 2$. The part (say Y) of M_n lying above such a compact sub-manifold of J_n is then a compact complex manifold, and one shows without difficulty that it is non-kählerian. On the other hand M_n is differentiably a product, and so one can put a kähler structure on Y using the kähler structure on the two factors.

"The complex manifold M_n is a 'fibre space of complex tori', and in some sense it is the most general space of this type. To make this concept precise we study fibre spaces of tori in general and then prove that M_n has a certain 'universal' property.

"The fibres in M_n all have a group structure, so that M_n is actually a 'fibre space of groups'. We can however consider fibre spaces of tori where the fibres are simply affine tori, i.e. without preferred base points. These give more examples of non-kähler manifolds, though now they are no longer differentiable products.

"Of the many ways in which the complex structure of J_n may be introduced we have chosen the method of embedding J_n in a complex Grassmannian. The advantages of this method for our purposes may be briefly listed: (a) the complex structure of M_n follows immediately from the complex structure of the universal vector bundle over the Grassmannian, (b) the 'universality' of M_n follows from that of the Grassmannian, (c) the existence of compact sub-manifolds becomes transparent, (d) the computation of Chern classes is simplified.

"The two different complex structures on Y have quite distinct Chern classes as we verify. We conclude by constructing a further class of examples, this time of simply-connected non-kähler manifolds. Unfortunately these examples do not settle the question, which therefore remains open, of whether a simply-connected compact differentiable manifold can carry both kählerian and non-kählerian structures." (Author's introduction)

E. Calabi (Minneapolis, Minn.)

4455:

Eum, Sang-Seup. On the semi-simple group space with a Kaehlerian metric. Kyungpook Math. J. **1** (1958), 49–56.

The author tries to develop the differential geometry of a compact semi-simple Lie group admitting a Kaehler metric invariant by both the right and the left translations. Among other things, the author proves that such a Kaehlerian metric is flat. But it seems to the reviewer that this conclusion simply shows the well-known fact that such a group does not exist. *K. Yano* (Hong Kong)

4456:

Eum, Sang-Seup. Bochner's lemma on the Craig extensor field. Kyungpook Math. J. **1** (1958), 75–80.

Bochner's lemma referred to here is the following: In a compact Riemannian space with a positive definite metric, if a function $\phi(x)$ satisfies

$$L(\phi) = g^{ij}\frac{\partial^2\phi}{\partial x^i \partial x^j} + h^i\frac{\partial\phi}{\partial x^i} \geq 0$$

everywhere in the space, then we have $\phi(x) = \text{const}$ everywhere in the space. The author tries to generalize this lemma to the case of the space with metric extensor g^{ijkl}. It seems to the reviewer that the most essential assumption in this lemma is the fact that g^{ij} is positive definite. But the present paper says nothing about the positive definiteness of the extensor g^{ijkl}. So the reviewer can say nothing about the validity of the theorems obtained in the present paper. *K. Yano* (Hong Kong)

4457:

Singh, H. D. On the contractions of extensors. Tensor (N.S.) **9** (1959), 94–98.

The contraction theory of extensors over full and reduced range has been established by H. V. Craig [*Vector and tensor analysis*, McGraw-Hill, New York, 1943; MR **5**, 77] and A. Kawaguchi [Proc. Imp. Acad. Tokyo **14** (1938), 237–241]. In this paper certain theorems on contraction of extensors are extended and generalized.

A. Kawaguchi (Sapporo)

4458:

★Efimow, N. W. Flächenverbiegung im Grossen. Mit einem Nachtrag von E. Rembs und K. P. Grotemeyer. Akademie-Verlag, Berlin, 1957. xi+233 pp. DM 33.50.

[The Russian original appeared in Uspehi Mat. Nauk (N.S.) **3** (1948), no. 2 (24), 47–158; MR **10**, 324; this translation was noted in MR **19**, 59.] When Cohn-Vossen went to Moscow in 1936 he took with him, so to say, the theory of bending of surfaces. Many important papers on the subject appeared in Russia in the following years, among them Efimov's review that forms the basis of the present book (which to an extent brings the theory back to Germany). To the 112 pages which contain the translation and 57 titles of bibliography are added 121 pages of appendices with well over one hundred references. These appendices round out the presentation of the original. They contain a wealth of material ranging historically from Cauchy's fundamental paper of 1813 to results previously not published. Methodologically they range from elementary geometry in the study of polyhedra to the use of the Poincaré index, the theory of homology groups, elaborate use of tensor calculus, in which simultaneous differential and integral invariants are introduced whose vanishing signifies essentially congruence of surfaces, theory of elliptic systems of differential equations, integral formulas first introduced, it seems, by Minkowski, projective transformations and line geometry. The book contains many new results, mostly refinements of those reviewed in the original; a large number of them are due to Rembs and Grotemeyer, the authors of the appendices. The exposition is very clear and detailed, although complicated proofs in some cases are omitted.

G. Y. Rainich (Notre Dame, Ind.)

4459:

Rembs, E. Flächen mit demselben sphärischen Bild. Monatsh. Math. **63** (1959), 209–213.

The author gives new proofs of the following results. Suppose corresponding points of the convex surfaces $\mathfrak{x} = \mathfrak{x}(u, v)$ and $\bar{\mathfrak{x}} = \bar{\mathfrak{x}}(u, v)$ have the same spherical image and the same Gauss curvature. Then $\mathfrak{x}$ and $\bar{\mathfrak{x}}$ are congruent by translation if they also satisfy any one of the following conditions: (i) they are closed [Th. 2]; (ii) (they are simply connected and) their boundaries are congruent by translation [Th. 3]; (iii) their boundaries lie in parallel planes [Th. 4]. Outline of the proofs: Put $\mathfrak{y} = \mathfrak{x} + \bar{\mathfrak{x}}$, $\bar{\mathfrak{y}} = \mathfrak{x} - \bar{\mathfrak{x}}$.

Then $\bar{\mathfrak{x}}_u \times \mathfrak{x}_v = \bar{\mathfrak{x}}_u \times \bar{\mathfrak{x}}_v$ implies $\bar{\mathfrak{y}}_u \times \mathfrak{y}_v = \bar{\mathfrak{y}}_v \times \mathfrak{y}_u$. Thus $\bar{\mathfrak{y}}$ determines an infinitesimal deformation of the convex surface $\mathfrak{y}$. But each of (i)—(iii) implies that $\mathfrak{y}$ is rigid. Thus $\bar{\mathfrak{y}} = \text{const.}$ *P. Scherk* (Toronto, Ont.)

4460:

Fuglede, Bent. An integral formula. Math. Scand. **6** (1958), 207–212.

Let $x, \cdots$ denote vectors in euclidean n-space R^n; $|x| =$ length of x; $dx = n$-dimensional Lebesgue measure; $w_m =$ area of the unit sphere in R^m. Let $G = \{g\}$ be the group of the rotations of R^n about the origin. It is compact under its standard topology. Let A^k be the set of the linear k-spaces $\alpha, \cdots$ through the origin of R^n. Let $\alpha_0 \in A^k$ be fixed; $H = \{g \mid g\alpha_0 = \alpha_0\}$. Then A^k can be identified with the homogeneous space G/H. Let μ be the image on G/H of the normalized Haar measure on G under the canonical mapping $G \to G/H$. Then μ will be the—unique—normalized G-invariant measure of $A^k = G/H$. Applying Fubini's theorem, the author proves

$$\int_{A^k} \left(\int_\alpha f(x) d\delta(x) \right) d\mu(\alpha) = \frac{w_k}{w_n} \int_{R^n} |x|^{k-n} f(x) dx.$$

Here f is any Baire function on R^n with values $0 \leq f(x) \leq +\infty$ and δ is the k-dimensional Lebesgue measure on α. His formula yields i.a. a new solution of Radon's problem: To determine a function in R^n if its integrals over the linear k-manifolds in R^n are given [Ber. Verh. Sächs. Akad. Wiss. Leipzig, Math. Nat. Kl. **69** (1917), 262–277]. *P. Scherk* (Toronto, Ont.)

4461:

Lippmann, Horst. Metrische Eigenschaften verschiedener Winkelmasse im Minkowski- und Finslerraum. I, II. Nederl. Akad. Wetensch. Proc. Ser. A **61** = Indag. Math. **20** (1958), 223–238.

The review MR **20** #4766 of this paper was by error printed in the wrong section.

4462:

★Rund, Hanno. The differential geometry of Finsler spaces. Die Grundlehren der Mathematischen Wissenschaften, Bd. 101. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1959. xiii + 284 pp. DM 59.60.

This is the first well-arranged book furnishing a reasonably comprehensive account of the new field, theory of Finsler spaces, developed by a large number of geometers very rapidly in the last 30 years. The method of tensor calculus is used, mainly, and it is intended that this book may serve also as an introduction to a branch of differential geometry which is closely related to various topics in theoretical physics, notably analytical dynamics and geometrical optics. There exist several fundamentally different points of view with regard to Finsler geometry, not only on the basis of the tensor calculus but also in other senses, e.g., that of Busemann [cf. *The geometry of geodesics*, Academic Press, New York, 1955; MR **17**, 779], which do not fall within the scope of this treatise. Moreover this book does not include an account of the more recent generalisations such as the geometries of Cartan spaces and of Kawaguchi spaces. As the author says in the preface, a book on general metric spaces is being planned by E. T. Davies which would deal with these last geometries. This book consists of introduction, six chapters and appendix. The introduction is devoted to a brief historical statement of the development of the theory. In Chapter I, the simplest problems in the calculus of variations are formulated in which the origin of the theory of Finsler spaces is to be found. While no previous knowledge of this topic is presupposed, no attempt is made to cover the ground, but only those developments which play a central role in the theory of Finsler spaces, such as the condition of Legendre, are discussed in detail, and as far as possible from a geometrical point of view. It is indicated how a problem in the calculus of variations imposes a metric on the underlying manifolds, and the local properties of such a metric are described very well by the introduction of the so-called tangent space. A large section of this chapter is devoted to the study of metric properties of the tangent spaces which will lead naturally to the definition of Minkowskian space. Finsler manifolds are, in fact, locally Minkowskian spaces. The definitions of the fundamental metric concepts, such as length, angle, trigonometric ratios and areas, are formulated in terms of a local Minkowskian geometry. In Chapter II the covariant differentiation, i.e. δ-differentiation in the sense of the author, of a vector is derived from the differential equations satisfied by the extremals, i.e. geodesics, where the geodesics are found by means of a method specially adopted to illustrate clearly the geometric background underlying this derivation. More precisely, conditions are established for geometrically meaningful mappings of one such tangent space onto another, i.e. the type of conditions which must be satisfied such that two vectors belonging to distinct but neighbouring tangent spaces may be described as being parallel. The well-known euclidean connection of E. Cartan is explained in Chapter III, which is based upon the idea of the so-called element of support, and the covariant derivatives of the metric tensor with respect to this connection vanish identically, i.e. Ricci's lemma is valid. In the latter half of this chapter Finsler spaces are generalised by defining a general space of paths. Chapter IV is devoted to curvature theory. The relevant commutation formulae give rise to the curvature tensors which satisfy a large number of identities. Spaces of constant curvature, projective curvature tensors and projective connection are also discussed. In Chapter V we find the theory of subspaces, which is treated from the two different aspects, viz., of the euclidean connection on the one hand and the locally Minkowskian theory on the other, from a more or less unified point of view. In Chapter VI there are miscellaneous topics related to the theory of Finsler spaces, e.g., groups of motions, conformal geometry, equivalence problem, theory of non-linear connections, local imbedding theories and two-dimensional Finsler spaces. The appendix contains bibliographical references to related topics: Recent survey articles, Finsler spaces with metric functions of special forms, non-holonomic spaces, infinite-dimensional Finsler spaces, Cartan spaces, Kawaguchi spaces and applications of the methods of metric differential geometry to theoretical physics. At the end of this book there is an almost complete list of publications dealing with Finsler spaces up to July, 1957. A knowledge of the techniques of the tensor calculus and linear algebra is presupposed, and while an acquaintance with Riemannian geometry is

highly desirable in view of frequent references to this subject, this requirement is not absolutely essential. The same applies to the classical differential geometry of curves and surfaces. *A. Kawaguchi* (Sapporo)

PROBABILITY

See also 4334, 4593, 4674, 4863, 4866, 4870.

4463:

★**Le Cam, L. Remarques sur les variables aléatoires dans les espaces vectoriels non séparables.** Le calcul des probabilités et ses applications. Paris, 15–20 juillet 1958, pp. 39–53. Colloques Internationaux du Centre National de la Recherche Scientifique, LXXXVII. Centre National de la Recherche Scientifique, Paris, 1959. 196 pp.

L'auteur se propose d'étendre aux espaces vectoriels non séparables certains théorèmes classiques relatifs aux variables réelles à une ou plusieurs dimensions. Ceci le conduit à des considérations d'ordre général, en particulier à une classification des mesures sur les espaces topologiques et à l'établissement de relations entre le comportement des mesures sur les espaces vectoriels généraux et celui de leurs transformées de Fourier.

Dans un dernier chapitre est traitée une extension du théorème de P. Lévy relatif à l'augmentation de la dispersion. *A. Fuchs* (Strasbourg)

4464:

Bertaut, Félix. Densité de probabilité. Positivité et ordre. C. R. Acad. Sci. Paris **249** (1959), 27–29.

Posons $E(x_1, \cdots, x_t) = \sum_{i=1}^{t} G(x_i)/\sqrt{t}$, les $G(x_i)$ étant des variables aléatoires indépendantes réduites (l'auteur ne fait pas la différence entre fonction et variable aléatoire). Alors la densité de probabilité $P(A)$ de E peut s'écrire en fonction des cumulants k_p et des polynômes d'Hermite $H_\theta(A)$ sous la forme de Charlier:

$$P(A) = (2\pi)^{-1/2} \exp(-\tfrac{1}{2}A^2)S,$$

où

$$S = 1 + \frac{k_3}{3!} H_3(A) + \frac{1}{2!}\left(\frac{k_3}{3!}\right)^2 H_6(A) + \cdots.$$

La série définissant S n'est pas positive partout et contient des polynômes de degrés différents associés à un même ordre de grandeur. L'auteur donne un procédé pour éviter ces inconvénients. *A. Fuchs* (Strasbourg)

4465:

Blum, J. R.; and Rosenblatt, M. On the structure of infinitely divisible distributions. Pacific J. Math. **9** (1959), 1–7.

A theorem is proved giving necessary and sufficient conditions for an infinitely divisible distribution on the real line to be discrete, or continuous, or a mixture. These conditions relate to the Khintchine representation of the characteristic function of such a distribution. The theorem is applied to characterize a class of discrete stochastic processes in the manner of Khintchine [Teor. Veroyatnost. i Primenen. **1** (1956), 3–18, 320–327; MR **19**, 328]. *K. Chandrasekharan* (Bombay)

4466:

Takács, L. On the probability distribution of the measure of the union of random sets placed in a Euclidean space. Ann. Univ. Sci. Budapest. Eötvös. Sect. Math. **1** (1958), 89–95.

The author determines for several cases of random placings of sets in E^m the moments of the volume of the union of the sets, from which the distribution can be deduced. Several authors have considered special cases in E^1. J. Bronowski and J. Neyman [Ann. Math. Statist. **16** (1945), 330–341; MR 8, 389] investigated squares in E^2 and H. Robbins [ibid. 342–347; MR 8, 389] considered rectangles and circles in E^2. *A. Dvoretzky* (Jerusalem)

4467:

Čèn', Pin'. The number of non-negative jump points of an empirical distribution function relative to a distribution function. Acta Math. Sinica **8** (1958), 333–347. (Chinese. Russian summary)

Let F_n be the n-sample empiric distribution function for the theoretical distribution F, assumed to be continuous. If $F_n(x+0) > F(x+0)$ and $F_n(x) < F_n(x+0)$, then x is a non-negative jump point. Let its number be τ_n. It is shown that

$$P\{\tau_n = h\} = \frac{n!}{n^n} \sum_{a=1}^{h} \frac{a^{a-1}(n-a)^{n-a}}{a!(n-a+1)!} \quad (h = 1, \cdots, n).$$

Asymptotic expansions are given for $P\{\tau_n \leq nt\}$ and $E(\tau_n)$.

K. L. Chung (Syracuse, N.Y.)

4468:

Šalaevskiĭ, O. V. Stability in Raĭkov's theorem. Vestnik Leningrad. Univ. **14** (1959), no. 7, 41–49. (Russian. English summary)

The author considers the problem of estimation of the distributions of two independent random variables X_1 and X_2 when the distribution of their sum $X = X_1 + X_2$ is only approximately Poisson; not strictly Poisson as in Raĭkov's theorem. A theorem is proved in which estimates of the component distributions are given.

H. P. Edmundson (Pacific Palisades, Calif.)

4469:

Laha, Radha Govinda. Une remarque sur les fonctions caractéristiques analytiques. C. R. Acad. Sci. Paris **249** (1959), 358–359.

Let $f(z)$ be an analytic characteristic function, that is, a characteristic function which coincides with a regular analytic function in some neighborhood of the origin. Denote its even part by $f_0(z) = [f(z) + f(-z)]/2$ and its odd part by $f_1(z) = [f(z) - f(-z)]/2$. Let ρ, ρ_0, ρ_1 be the radii of convergence of $f(z)$, $f_0(z)$, $f_1(z)$ respectively about the point $z = 0$. It is known that $\rho_1 \geqq \rho_0 \geqq \rho$. The author shows that the stronger relation $\rho_1 = \rho_0 = \rho$ holds for every analytic characteristic function.

E. Lukacs (Washington, D.C.)

4470:

Méric, Jean. Sur un problème de marche au hasard dans le plan. Distribution du nombre de pas. C. R. Acad. Sci. Paris **248** (1959), 3111–3113.

"Le problème abordé est une généralisation de la marche au hasard déjà étudiée en liaison avec le test de Wald. Le présent travail établit, pour la variable aléatoire:

nombre de pas, la fonction génératrice des moments. Ce résultat est suivi de quelques exemples simples." (Author's summary) *J. Wolfowitz* (Ithaca, N.Y.)

4471:

Le Cam, Lucien. Un théorème sur la division d'un intervalle par des points pris au hasard. Publ. Inst. Statist. Univ. Paris **7** (1958), no. 3/4, 7–16.

Étant donné un processus de Poisson, on sait que, si dans un intervalle donné il y a n tops, ces tops sont distribués indépendamment les uns des autres et uniformément sur l'intervalle considéré. L'auteur étudie dans le présent article une intéressante réciproque de ce théorème.

Pour tout entier $n>1$, soient $X_{n,i}$ $(i=1, \cdots, n-1)$ des variables aléatoires indépendantes uniformément réparties sur $[0, 1]$; soient $X^*_{n,i}$ $(i=1, \cdots, n-1)$ les variables aléatoires ordonnées par ordre de grandeur croissante qui correspondent aux $X_{n,i}$. Posons $X^*_{n,0}=0$, $X^*_{n,n}=1$ et $U_{n,i}=n[X^*_{n,i}-X^*_{n,i-1}]$. Soit $\{g_n\}$ une suite de fonctions de Baire définies sur $[0, \infty[$.

Soit d'autre part $\{V_i\}$ $(i=1, 2, \cdots)$ une suite de variables aléatoires indépendantes de densité e^{-x}.

Alors pour n grand la distribution de $T_n=\sum_{i=1}^{n} g_n(V_i)$ est voisine de celle de $W_n=\sum_{i=1}^{n} g_n(U_{n,i})$ en un sens que l'auteur précise. *A. Fuchs* (Strasbourg)

4472:

★**Whittle, P. Sur la distribution du maximum d'un polynome trigonométrique à coefficients aléatoires.** Le calcul des probabilités et ses applications. Paris, 15–20 juillet 1958, pp. 173–184. Colloques Internationaux du Centre National de la Recherche Scientifique, LXXXVII. Centre National de la Recherche Scientifique, Paris, 1959. 196 pp.

Let

$$C_N(\omega) = 2N^{-1/2} \sum_{k=1}^{n} X_k \cos k\omega$$

be a random trigonometric polynomial whose coefficients X_k are independent identically distributed chance variables with a finite absolute moment of order $s>4$. Then for any $\varepsilon>0$,

$$\lim_{N\to\infty} P\{[2(1-\varepsilon)\log N]^{1/2} \leq \max_\omega |C_N(\omega)| \leq [2(1+\varepsilon)\log N]^{1/2}\} = 1.$$

This overlaps a result of Salem and Zygmund [Acta Math. **91** (1954), 245–301; MR **16**, 467]. Extensions are discussed. *H. Teicher* (Lafayette, Ind.)

4473:

Prohorov, Yu. V. An extremal problem in probability theory. Teor. Veroyatnost. i Primenen. **4** (1959), 211–214. (Russian. English summary)

Let $X_1, \cdots, X_n$ be bounded, independent random variables; let $E(X_k)=0$ and $\Pr\{|X_k|\leq c\}=1$ for $k=1, 2, \cdots, n$, and $\sigma^2=E(X_1+\cdots+X_n)^2$. Then it is shown that

$$\Pr\{X_1+\cdots+X_n \geq x\} \leq \exp\left\{-\left(\frac{x}{2c}\right)\sinh^{-1}\left(\frac{xc}{2\sigma^2}\right)\right\}$$

if $x\geq 0$. By an oversight, the restriction $x\geq 0$, which is essential, is missing from the statement of the theorem. An example is given to claim a certain degree of sharpness for the order of the bound. Upper and lower bounds are also given for $\Pr\{Y\geq y\}$, where Y is the difference of two identically distributed, independent Poisson random variables. *S. G. Ghurye* (Evanston, Ill.)

4474:

Siraždinov, S. H. On exact estimation in a local limit theorem. Teor. Veroyatnost. i Primenen. **4** (1959), 229–233. (Russian. English summary)

Let $\{\xi_n\}$ be a sequence of independent, identically distributed random variables with common probability density $p(x)$, $\int xp(x)dx=0$, $\int x^2p(x)dx=1$. Let $p_n(x)$ denote the probability density of $n^{-1/2}(\xi_1+\cdots+\xi_n)$, $\phi(x)=(2\pi)^{-1/2}\exp(-x^2/2)$, $C_n^{(p)}=\int |p_n(x)-\phi(x)|^p dx$. Theorem: If $\alpha_3=\int x^3p(x)dx$ exists and, for some n,

$$p_n(x) \in L_\alpha(-\infty, \infty),$$

where $1<\alpha\leq 2$, then for any $p\geq 1$

$$C_n^{(p)} = \lambda_p(|\alpha_3|n^{-1/2})^p + o(n^{-p/2}),$$

where $\lambda_p=6^{-p}\int |3x-x^3|^p[\phi(x)]^p dx$.

W. Hoeffding (Chapel Hill, N.C.)

4475:

Spitzer, Frank. Some probability limit theorems. Bull. Amer. Math. Soc. **65** (1959), 117–119.

Announcement of results whose proofs will appear elsewhere. Let $X_1, X_2, \cdots$ be independent and identically distributed with $EX_i=0$, $EX_i^2=1$, and $S_k=X_1+\cdots+X_k$. Theorem 1: $\sum_k k^{-1}(\frac{1}{2}-P\{S_k>0\})$ converges (conditionally), say to $\log c$. Theorem 2: E (first positive S_i) $=2^{-1/2}c$. Theorem 3: $\lim_n \Pr\{S_1>0, \cdots, S_n>0\}(n\pi)^{1/2}=c^{-1}$. Theorem 4: $[k(n-k)]^{1/2}\Pr\{\text{number of positive } S_i\ (1\leq i\leq n)=k\}=\pi^{-1}+o(1)$ as $\min(k, n-k)\to\infty$. Theorem 5: $\lim_n E\{n^{-1/2}S_n|S_1\geq 0, \cdots, S_n\geq 0\}=(\pi/2)^{1/2}$. Theorem 6: If the X_i are also symmetric aperiodic lattice random variables, r is an integer, and N_{A2} [resp., N_{A1}] = number of k for which $S_k=r$ and $S_i\leq A$ [resp., $|S_i|\leq A$] for $i\leq k$, then, for $x>0$, $\lim_{A\to\infty}\Pr\{N_{Aj}>jAx\}=e^{-x}$.

J. Kiefer (Ithaca, N.Y.)

4476:

Prohorov, Yu. V. Some remarks on the strong law of large numbers. Teor. Veroyatnost. i Primenen. **4** (1959), 215–220. (Russian. English summary)

Let $\{X_n, n=1, 2, \cdots\}$ be a sequence of individually bounded, independent random variables, with $P\{|X_n|\leq c_n\}=1$, $n=1, 2, \cdots$, for a certain sequence $\{c_n\}$ of positive constants. The necessary and sufficient conditions for the strong law of large numbers for independent random variables, given earlier by the author [Izv. Akad. Nauk SSSR. Ser. Mat. **14** (1950), 523–536; MR **12**, 425], are discussed for this special case. It is shown that they can be expressed in terms of the variances of X_n if $c_n=O(n/\log\log n)$. The case $(\log\log n)c_n/n\to\infty$ is also discussed. *S. G. Ghurye* (Evanston, Ill.)

4477:

Volkonskiĭ, V. A.; and Rozanov, Yu. A. Some limit theorems for random functions. I. Teor. Veroyatnost. i Primenen. **4** (1959), 186–207. (Russian. English summary)

Let $\mathscr{X}(\Delta)$ be a random function defined on the (one-dimensional) interval Δ. In this paper the authors prove nine limit theorems for random functions $\mathscr{X}(\Delta)$ which satisfy the strong mixing condition

$$\lim_{\tau\to 0} \sup_{t} \sup_{\substack{A\in\mathfrak{M}_{-\infty}^{t} \\ B\in\mathfrak{M}_{t+\tau}^{\infty}}} |\mathbf{P}(AB)-\mathbf{P}(A)\mathbf{P}(B)| = 0.$$

In the above $\mathfrak{M}_{t_1}^{t_2}$ is a σ-algebra of subsets of the probability space generated by events of the form $\{\mathscr{X}(\Delta_1)<h_1, \cdots, \mathscr{X}(\Delta_n)<h_n\}$, where $\Delta_1, \cdots, \Delta_n$ are arbitrary intervals, with $\Delta_i\subseteq(t_1, t_2)$, and $h_1, \cdots, h_n$ are arbitrary real numbers. The lemmas and theorems established are too long and involved to reproduce here.

A. T. Bharucha-Reid (Eugene, Ore.)

4478:

Révész, P. A limit distribution theorem for sums of dependent random variables. Acta Math. Acad. Sci. Hungar. **10** (1959), 125–131. (Russian summary, unbound insert)

Applying a theorem of Rényi [same Acta. **9** (1958), 215–228; MR **20** #4623] the author proves that under certain conditions, the sums

$$\zeta_n = \xi_1+\xi_2+\cdots+\xi_n$$

and

$$\zeta_n^* = \xi_1^*+\xi_2^*+\cdots+\xi_n^*$$

concerning the sequences of random variables $\xi_1, \xi_2, \cdots, \xi_n, \cdots$ and $\xi_1^*, \xi_2^*, \cdots, \xi_n^*, \cdots$—the latter being independent amongst each other—are asymptotically equivalent in distribution. These conditions bear upon the relations between the multivariate distributions

$$P(\xi_1<x_1, \xi_2<x_2, \cdots, \xi_n<x_n), \quad P(\xi_1^*<x_1, \cdots, \xi_n^*<x_n)$$

of the two sequences, respectively, and characterize an almost independency of the variables of the sequence $\xi_1, \xi_2, \cdots, \xi_n, \cdots$.

O. Onicescu (Bucharest)

4479:

Nelson, Edward. Regular probability measures on function space. Ann. of Math. (2) **69** (1959), 630–643.

Soit X un espace compact, T un ensemble quelconque et Ω l'ensemble des applications de T dans X, avec la topologie produit. L'A. appelle "version canonique d'un processus stochastique" la réalisation du processus dont les événements sont les ensembles boréliens de Ω. L'existence de cette version est démontrée en utilisant la représentation des fonctionnelles linéaires et le théorème de Weierstrass-Stone. Ensuite on démontre la mesurabilité de certains ensembles de Ω. *G. Marinescu* (Bucharest)

4480:

Blumenthal, R. M. Some relationships involving subordination. Proc. Amer. Math. Soc. **10** (1959), 502–510.

By making use of a certain relation of subordination, the author investigates properties of the process $g(X(t))$, where $X(t)$ is a stochastic process taking values in a space $\mathfrak{X}$ and g a real-valued function defined on $\mathfrak{X}$. Moreover, under certain conditions involving the processes $X(t)$ and $g(X(t))$, some properties concerning g are deduced.

R. Theodorescu (Bucharest)

4481:

★Krickeberg, Klaus. Semi-martingales à base filtrante décroissante. Le calcul des probabilités et ses applications. Paris, 15–20 juillet 1958, pp. 133–138. Colloques Internationaux du Centre National de la Recherche Scientifique LXXXVII. Centre National de la Recherche Scientifique, Paris, 1959. 196 pp.

The author recalls the definitions of martingales and semi-martingales, the set of indices being an arbitrary directed set. In addition to the usual notions of strong convergence and stochastic convergence, he also deals with "essential convergence": in the ordered set of classes of measurable functions, any directed set has an infimum and a supremum, hence it is also possible to define lim sup and lim inf in that set, and the coincidence of these two limits defines essential convergence. The author states without proof a certain number of convergence theorems, for these various types of convergence; only very mild assumptions are needed for the two first types, but for essential convergence it is necessary to introduce additional restrictions, and the author states that conditions of the Vitali type are sufficient. *J. Dieudonné* (Paris)

4482:

Béthoux, Paul. Filtrage d'une fonction aléatoire dont la moyenne est une fonction linéaire. C. R. Acad. Sci. Paris **248** (1959), 3685–3686.

Considérons les fonctions aléatoires $Y(t)=a+bt+X(t)$ et $Y_1(t)=a+bt+X_1(t)$, a et b étant des paramètres inconnus, $X(t)$ et $X_1(t)$ étant des fonctions aléatoires du second ordre de moyenne nulle et stationnairement corrélées entre elles. L'auteur étudie la "meilleure" estimation de $Y(t_0+h)$, $h>0$, à partir des observations de $Y_1(s)$ pour $s\in[t_0-T, t_0]$. *A. Fuchs* (Strasbourg)

4483:

Moran, P. A. P. The survival of a mutant gene under selection. J. Austral. Math. Soc. **1** (1959/61), part 1, 121–126.

A population of $2n$ haploid individuals, of which initially k_0 are mutant, maintains constant size. A death is immediately preceded by a birth to some individual. Two kinds of selection are considered: (i) the life-times of mutants and non-mutants are negative-exponential with means λ_1 and λ_2, respectively, but the individual giving birth is chosen at random; (ii) $\lambda_1=\lambda_2$ but the odds that the birth is to a mutant are $\mu_1 k/\mu_2(2n-k)$, where $\mu_1\neq\mu_2$ and k is the number of mutants. Using an embedded Markov process, the probability of mutant extinction is shown to be the probability of gambler's ruin (gambler's capital, k_0; bank capital, $2n-k_0$; equal stakes; odds λ_1/λ_2 or μ_1/μ_2). If α (either $(\lambda_1-\lambda_2)/\lambda_2$ or $(\mu_1-\mu_2)/\mu_2$) is such that αk_0 is small and positive, the probability is $\alpha k_0/[1-\exp(-2\alpha n)]$. *M. Stone* (Cambridge, England)

4484:

Gani, J.; and Prabhu, N. U. The time-dependent solution for a storage model with Poisson input. J. Math. Mech. **8** (1959), 653–663.

Denote by $Z(t)$ the content of an infinite dam at the instant t and let $Z(0)=z_0$. The dam is fed by impulses of magnitude h (constant) at random instants which form a

homogeneous Poisson process with density λ. There is a continuous release with slope -1 in a non-empty dam. The authors determine by combinatorial methods the probability that the dam becomes empty for the first time at the instant $t=z_0+rh$ $(r=0, 1, 2, \cdots)$ and give the explicit form of the distribution function

$$P\{Z(t) \leqq z \mid Z(0) = z_0\} = F(z,t)$$

which is a solution of the partial differential-difference equation

$$\frac{\partial F(z,t)}{\partial z} - \frac{\partial F(z,t)}{\partial t} = -\lambda[F(z,t) - F(z-h,t)].$$

L. Takács (New York, N.Y.)

4485:

Prabhu, N. U. Application of generating functions to a problem in finite dam theory. J. Austral. Math. Soc. **1** (1959/61), part 1, 116–120.

The reservoir model in question is one of finite storage capacity whose annual increments are independent discrete random variables, each with the same probability distribution, and whose decrements, taken only at year-end, are either unity or the amount in storage, whichever is smaller. The stationary distribution of the amount in storage is determined in generating function form, by a procedure similar to that of N. T. J. Bailey [J. Roy Statist. Soc. Ser. B. **16** (1954), 80–87; MR **16**, 148]. Explicit results are given for two cases, namely (i) the distribution of the increments is of geometric type, (ii) the same distribution is of negative binomial type.

J. Riordan (New York, N.Y.)

4486:

Sakino, Sigeki; and Hayashi, Chikio. On the analysis of epidemic model. I. (Theoretical approach). Ann. Inst. Statist. Math. Tokyo. **10** (1959), 261–275.

The authors study stochastic epidemics in closed populations and make allowance for subclinical immunization (i.e., direct passage from the "susceptible" to the "recovered" class). For the associated deterministic model the differential equations take the form

$$(*) \qquad \begin{cases} Dx = -(\beta_1+\beta_2)xy, \\ Dy = \beta_1 xy - \beta_3 y, \\ Dz = \beta_2 xy + \beta_3 y, \end{cases}$$

where x, y, and z are respectively the numbers in the "susceptible", "infectious", and "recovered" classes. But the authors do not study (*) and do not even mention the threshold phenomenon. They have computed 10 artificial stochastic epidemics by a procedure approximately equivalent to the stochastic analogue to (*), and present graphs showing the variation of (1) $\xi(X)$, (2) $\xi(Y)$, (3) var (X), (4) var (Y), and (5) cov (X, Y) as functions of the time t. (Here X and Y are the random variables corresponding to x and y.)

The authors comment on the imperfections of current stochastic epidemic models, and stress particularly the desirability of making the chance of recovery in $(t, t+dt)$ depend in a realistic way on the time since infection. In a later paper they promise to construct such a model, and briefly sketch what they propose to do. {To the reviewer it would seem more natural to achieve the desired result by allowing several successive independently negative-exponentially distributed phases in the recovery process.}

The paper concludes with some deterministic calculations.

D. G. Kendall (Oxford)

4487:

★Hammersley, J. M. Bornes supérieures de la probabilité critique dans un processus de filtration. Le calcul des probabilités et ses applications. Paris, 15–20 juillet 1958, pp. 17–37. Colloques Internationaux du Centre National de la Recherche Scientifique, LXXXVII. Centre National de la Recherche Scientifique, Paris, 1959. 196 pp.

Ein Filtrationsprozess ist ein "percolation process" wie vom Verfasser früher beschrieben [Broadbent und Hammersley, Proc. Cambridge Philos. Soc. **53** (1957), 629–641; MR **19**, 989; und Hammersley, ibid., 642–645; MR **19**, 989]. Als mathematisches Modell des "zufälligen Mediums", in dem die "Strömung" stattfindet, dient ein Graph, dessen Kanten teils orientiert, teils nicht orientiert sein können, wobei von jedem "Atom" (jeder Ecke) nur endlich viele Kanten ausgehen und jede nicht leere endliche Menge von Atomen ein Atom enthält, von dem eine Kante zu einem nicht dazu gehörigen Atom führt. Die Strömung wird als längs der Kanten vor sich gehend gedacht, bei orientierten Kanten nur in der so bestimmten Richtung, mit den Atomen als "Quellen". Dabei ist jede Kante mit der für alle Kanten gleichen Wahrscheinlichkeit q gesperrt und mit der Wahrscheinlichkeit $p=1-q$ geöffnet. Die kritische Wahrscheinlichkeit $p_d(A)$ eines Atoms A ist die obere Grenze der Zahlen p derart, daß eine von A als einziger Quelle ausgehende Strömung fast sicher nur endlich viele Atome erreicht. In einer früheren Arbeit [Ann. Math. Statist. **28** (1957), 790–795; MR **21** #374] hatte der Verfasser untere Schranken für $p_d(A)$ angegeben und nun leitet er obere Schranken ab. In einer Reihe von Beispielen und Spezialfällen (gewisse Kristalle, multiplikative Prozesse, nicht orientiertes planares Medium) werden diese explizit berechnet und näher untersucht.

K. Krickeberg (Aarhus)

4488:

Akaike, Hirotugu. On the statistical control of the gap process. Ann. Inst. Statist. Math. Tokyo. **10** (1959), 233–259.

The author investigates further the second of the stochastic processes considered in a previous paper [same Ann. **8** (1956), 87–94; MR **19**, 327]. In this (with the notation of the review cited) the random variables $x_n = x_n(\omega)$ $(n=0, \pm 1, \pm 2, \cdots)$ take only the values 0 and 1, and if $x_n(\omega)=1$ the probability that m is the least positive integer with $x_{n+m}(\omega)=1$ is p_m: it is assumed that $1/P = \sum mp_m < \infty$. The author now studies the distribution of $\nu_t(h) = x_{t+1} + \cdots + x_{t+h}$. This sum may be described, alternatively, as the number of renewals at (integral) times between t and $t+h+1$ of an article with life-distribution $\{p_m\}$ [cf. J. G. Skellam and L. R. Shenton, J. Roy. Statist. Soc. Ser. B **19** (1957), 64–111; MR **19**, 990; especially § 27]. He finds that, for $k \geqq 1$, the kth order factorial moment of $\nu_t(h)$ equals the coefficient of z^h in $k!Pz(1-z)^{-2} \times [p(z)/\{1-p(z)\}]^{k-1}$. In particular, this yields an expression for the variance $V(h)$ of $\nu_t(h)$. Suppose that estimates $\hat{p}_m$ of the p_m's are obtained from N independent observations of the "gaps", and that $\hat{V}(h)$ is the estimate of $V(h)$ calculated from $\{\hat{p}_m\}$. Assuming merely that $\sum m^2 p_m < \infty$ he gives a proof that $\{\hat{V}(h) - V(h)\}/\sqrt{N}$ is asymptotically

normally distributed with zero mean and variance D_h^2, where D_h^2 is given explicitly by a formula too elaborate for reproduction here. Under the stronger assumption that $\sum p_m z^m$ has a radius of convergence exceeding 1 he finds, for large h, the asymptotic value $\bar{V}(h)$ of $V(h)$, and that of the large-sample variance $\bar{\bar{V}}(h)$. He indicates, with illustrations drawn from sampling experiments, a way in which the applicability of his asymptotic formulae can be checked by suitably arranged samples, and applies his procedure to construct a control-scheme for certain production-processes such as silk-winding.

H. P. Mulholland (Exeter)

4489:

★**Bendat, Julius S. Principles and applications of random noise theory.** John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London; 1958. xxi+431 pp. $11.00.

This work is an attempt by a mathematician to present the principles of noise theory and its applications in "... understandable engineering language, emphasizing physical meanings and mathematical restrictions." Contrary to what one might expect, the level of rigor is not as high as in the recent engineering texts on the same subject by Laning and Battin [*Random processes in automatic control*, Wiley, New York, 1956; MR **18**, 74] and Davenport and Root [*An introduction to the theory of random signals and noise*, McGraw-Hill, New York-Toronto-London, 1958; MR **19**, 1090]. Indeed, there are numerous instances of sloppiness in the definitions of basic terms, the statements of restrictive assumptions, and in the interpretations of results. Furthermore, there are many misleading statements and some errors. To cite an example, the author asserts (p. 114) that if $Y=f(X)$, X being a real-valued random variable with density function $p(x)$, and to each value of Y correspond n values of X, then the density of Y is given by $q(y)=[np(x)]/f'(x)$, with x replaced by its "equivalent" y.

The first three chapters present the fundamentals of noise theory, with the notions of ergodic process and probability introduced in Chapters 1 and 3, respectively. Wiener's technique and some of its extensions are treated in Chapter 4. Chapters 5 through 10 are concerned with more specialized topics such as errors in autocorrelation measurements, envelope detection, analog simulation of time-varying systems, the prediction of non-stationary processes, and the zero-crossing problem.

The book does contain worthwhile material, but the shortcomings cited above detract from its usefulness.

L. A. Zadeh (Berkeley, Calif.)

STATISTICS

See also 4488, 4531, 4837, 4852, 4859.

4490:

Khatri, C. G. On conditions for the forms of the type XAX' to be distributed independently or to obey Wishart distribution. Calcutta Statist. Assoc. Bull. **8** (1959), 162–168.

Let X be a random matrix whose n columns are n independent normally distributed p-vectors, each with common covariance matrix Σ. Let A, B be $n \times n$ symmetric matrices. Theorem 1: Suppose rank $A \geq P$. XAX' has (non-central) Wishart distribution if and only if A is idempotent. Theorem 2: XAX', XBX' are independent if and only if $AB=0$. Theorem 3 is a version of theorem 2 for n matrices $A_1, \cdots, A_n$, connecting independence with a number of other properties.

M. Dwass (Evanston, Ill.)

4491:

Ostle, Bernard; and Steck, George P. Correlation between sample means and sample ranges. J. Amer. Statist. Assoc. **54** (1959), 465–471.

The authors prove that the symmetry of a population distribution function implies that the sample mean and the sample range are uncorrelated. They also construct an example to show that the converse is not true. This reviewer wishes to remark that a special case of the yet unproven converse of Daly's theorem [Ann. Math. Statist. **17** (1946), 71–74; MR **7**, 464] is erroneously ascribed to E. Lukacs and E. P. King [ibid **25** (1945), 389–394; MR **15**, 969] while it is due to V. Paskevich [Prikl. Mat. Meh. **17** (1956), 513–516; MR **17**, 53].

E. Lukacs (Washington, D.C.)

4492:

Isii, Keiiti. On a method for generalizations of Tchebycheff's inequality. Ann. Inst. Statist. Math. Tokyo **10** (1959), 65–88.

Let $\{\mu_0, \mu_1, \cdots, \mu_{2n}\}$ be a given set of (non-degenerate) moments of a random variable X; T be a given closed one-dimensional set. This paper is concerned with a study of Chebyshev-type inequalities, i.e. bounds for $\Pr\{X \in T\} \equiv P(T)$. The main theorem asserts that the best possible bounds are given by $L(T) \leq P(T) \leq U(T)$, where $L(T) = \sup Ef$, $U(T) = \inf Ef$, $f(x) = \sum_0^{2n} a_i x^i$, the inf and sup are taken all polynomials of degree $\leq 2n$ satisfying $f(x) \geq \chi_T(x)$ and $f(x) \leq \chi_T(x)$, respectively, for all real x; $\chi_T(x)$ is the indicator of T. This method has been used repeatedly, e.g., by P. O. Berge [Skand. Aktuarietidskr. **15** (1932), 65–77]. The case where T is open, as well as the bivariate case, is also discussed. Theorems dealing with the form of the extremal distributions are given, e.g., the number of points at which mass has to be placed in order to achieve equality. This bears on the work of Dresher [Duke Math. J. **20** (1953), 261–271; MR **14**, 1067] using moment spaces, and of Royden [Ann. Math. Statist. **24** (1953), 361–376; MR **15**, 107]. As examples, the author obtains an upper bound for $\Pr\{|X| \geq k\}$ when $\mu_1=0$, $\mu_2=1$, $\mu_3 \geq 0$, and μ_4 is given; and a lower bound for $P\{(X_1, X_2) \in T\}$, where T is an open square with vertices $(-a, -a)$, (b, b), the moments $\mu_{01}, \mu_{10}, \mu_{20}, \mu_{11}, \mu_{02}$ are given with $(\mu_{10}, \mu_{01}) \in T$.

I. Olkin (East Lansing, Mich.)

4493:

Boyd, A. V. The harmonic mean of independent $\gamma(\frac{1}{2})$ variates. Simon Stevin **33** (1959), 34–37.

4494:

Basu, D. On statistics independent of sufficient statistics. Sankhyā **20** (1958), 223–226.

An earlier paper by the same author [Sankhyā **15** (1955), 377–380; MR **17**, 640] contains the following theorem: Let $\{P_\theta\}$, θ in Ω, be probability measures on a sample

space, and let the statistic T be sufficient. If T_1, T are statistically independent, then the distribution of T_1 does not depend on θ. The present paper gives a counter-example to this theorem, but also provides some mild restrictions under which the theorem remains correct.

M. Dwass (Evanston, Ill.)

4495:

Raja Rao, B. On the relative efficiencies of BAN estimates based on doubly truncated and censored samples. Proc. Nat. Inst. Sci. India. Part A **24** (1958), 366–376.

Let $a<a_1<b_1<b$ and let $W(\theta)$ stand for a random variable $a \leqq W(\theta) \leqq b$ depending upon a parameter $\theta \in \Theta$, where Θ is an open set. The problem considered is that of precision in estimating θ as measured by the asymptotic variance of a BAN estimate based on n independent observation on $W(\theta)$ and, alternatively, on the same number n of independent observations made on each of three other variables $X(\theta)$, $Y(\theta)$ and $Z(\theta)$ defined as follows. If $a_1 \leqq W(\theta) \leqq b_1$ then the values of the three variables coincide, $X(\theta)=Y(\theta)=Z(\theta)=W(\theta)$. In other cases $X(\theta)$ is not defined and $Y(\theta)=b$. If $W(\theta)<a_1$ then $Z(\theta)=a$. If $W(\theta)>b_1$, then $Z(\theta)=b$. Under certain conditions of regularity it is shown that: (i) the precision of estimating θ using X may be worse than, equal to or better than that using W; (ii) the precision of estimating θ using either Y or Z cannot be better than that using W. The conditions imposed by the author appear unnecessarily strong.

J. Neyman (Berkeley, Calif.)

4496:

Wijsman, Robert A. On the theory of BAN estimates. Ann. Math. Statist. **30** (1959), 185–191.

The paper gives a generalization of the concept of the BAN estimates [Neyman, *Proceedings of the Berkeley Symposium on Mathematical Statistics and Probability, 1945, 1946*, pp. 239–273, Univ. California Press, Berkeley-Los Angeles, 1949; MR **10**, 383; Barankin and Gurland, Univ. California Publ. Statist. **1** (1950), 89–130; MR, **13**, 53; Chiang, Ann. Math. Statist. **27** (1956), 336–351; MR **19**, 694; Ferguson, ibid. **29** (1958), 1046–1062; MR **20** #6751]. The new definition omits a number of requirements of regularity originally adopted for convenience in proofs. The asymptotic properties of the new BAN estimates are the same as those of the old. Every old BAN estimate satisfies the new definition but not vice versa. The paper is inspired by the ideas of Le Cam [*Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954, 1955, vol. I*, pp. 129–156, Univ. California Press, Berkeley-Los Angeles, 1956; MR **18**, 947]. The interest of the paper would have been increased by the inclusion of an illustrative example.

J. Neyman (Berkeley, Calif.)

4497:

Kish, Leslie; and Hess, Irene. On variances of ratios and their differences in multi-stage samples. J. Amer. Statist. Assoc. **54** (1959), 416–446.

"Approximate estimators" are given for: (1) the ratio of totals for two measurable characteristics obtained from the same stratified sample, (2) the difference between two such ratios. Special cases considered are: $k(\geqq 2)$ primary selections taken from each stratum; systematic selection of primary selections. Modifications in procedure which result from variation of sampling technique—such as selecting primary sampling units with or without replacement, or selecting only one primary sample per stratum—are discussed. The precise meaning of "approximate estimator" of the variance is not made clear. The reviewer infers that either the approximations are to unbiased estimates of the variance or to unbiased estimates of the first few terms of a well known expansion for the variance. In either case, means for determining the degree of closeness of the approximation are not given.

M. Skibinsky (Lafayette, Ind.)

4498:

Schutzenberger, Marcel Paul. A propos de l'inégalité de Fréchet-Cramér. Publ. Inst. Statist. Univ. Paris **7** (1958), no. 3/4, 3–6.

Die genannte Ungleichung wird auf den Fall der sogenannten Bayesschen Schätzung ausgedehnt, wo der unbekannte Parameter x eine a priori Wahrscheinlichkeitsdichte $f(x)$ hat. Im Nenner tritt dann zusätzlich der Term $E_x((f'(x)/f(x))^2)$ auf (in der Definition des anderen Terms im Nenner ist übrigens ein Druckfehler enthalten). Bedingungen für das Gleichheitszeichen werden zunächst im allgemeinen Fall aufgestellt und dann in dem Spezialfall, dass jede der Beobachtungsvariablen y Summe einer sicheren (analytischen) Funktion $a(x)$ und einer von x unabhängigen kontinuierlich verteilten Variablen ist; in diesem Fall muss die gemeinsame Verteilung von x und y normal sein.

K. Krickeberg (Aarhus)

4499:

Fend, A. V. On the attainment of Cramér-Rao and Bhattacharyya bounds for the variance of an estimate. Ann. Math. Statist. **30** (1959), 381–388.

"If a variable X has density function $f(x, \theta)$, then in many cases the Cramér-Rao bound or the Bhattacharyya bounds may be used to show that a function $d(x)$ is a uniformly minimum variance unbiased estimate of the real parameter θ.

In this paper it is shown that if $f(x, \theta)$ is a member of the family of densities of the Darmois-Koopman form, and if the variance of $d(x)$ achieves the kth Bhattacharyya bound, but not the $(k-1)$th bound, then

$$f(x, \theta) = \exp[t(x)g(\theta)+g_0(\theta)+h(x)]$$

and $d(x)$ is a polynomial in $t(x)$ of degree k. Further, the variance of any polynomial in $t(x)$ of degree k will achieve the kth bound, so that if any such unbiased polynomial exists, it will necessarily be uniformly minimum variance unbiased. Some properties of these polynomial estimates are discussed." (Author's summary)

L. J. Savage (Chicago, Ill.)

4500:

★Radhakrishna Rao, C. Sur une caractérisation de la distribution normale établie d'après une propriété optimum des estimations linéaires. Le calcul des probabilités et ses applications. Paris, 15–20 juillet 1958, pp. 165–172. Colloques Internationaux du Centre National de la Recherche Scientifique, LXXXVII. Centre National de la Recherche Scientifique, Paris, 1959. 196 pp.

It is well known that for a normal population the least square estimate of the parameters of a linear hypothesis is a minimum variance estimate. The author proves the

converse statement: The assumption that the least square estimate of the parameters of a linear hypothesis is a minimum variance estimate implies (under certain restrictions) the normality of the population. Thus he obtains an interesting, new characterization of the normal population. *E. Lukacs* (Washington, D.C.)

4501:

Chatterjee, Shoutir Kishore. On an extension of Stein's two-sample procedure to the multi-normal problem. Calcutta Statist. Assoc. Bull. **8** (1959), 121–148.

A generalization to a p-variate normal universe $N(\mu, \Sigma)$ of the univariate two-sample procedure of C. Stein [Ann. Math. Statist. **16** (1945), 243–258; MR **7**, 213] based on the $p \times p$ covariance matrix S of the first $n_0 > p$ of the n variates sampled. The proposed statistic $u = z\Xi'\alpha\Xi$ involves an arbitrary number $z > 0$ and an arbitrary positive definite $p \times p$ matrix $\alpha = (\alpha_{rs})$, as well as a p-vector Ξ which depends on n_0, S and on all the n sample variates, and which must satisfy three conditions given by the author. The overall sample size is then $n = \max\{n_0 + p^2, [z\alpha S] + 1\}$. The power of the u-test of the hypothesis $\mu = \mu_0$, which is independent of Σ, is an increasing function of $\Delta = z(\mu - \mu_0)'\alpha(\mu - \mu_0)$ and increases with z at the cost of larger n. The choice of α fixes the "equidetectability contours" of the ellipsoids $(\mu - \mu_0)'\alpha(\mu - \mu_0) = k$, $0 < k < \infty$; $\alpha = I$, the identity matrix, when the p differences $\mu - \mu_0$ are equally important. Confidence regions of size $1 - \varepsilon$ of the form $(\mu - \Xi)'\alpha(\mu - \Xi) \leq u_{1-\varepsilon}/z$ follow immediately. The power function of the u-test is derived implicitly for $p > 2$ and explicitly for $p = 2$. The author promises an improvement of the foregoing test and confidence region on the same lines as Stein's improvement of his original results [loc. cit.]. *H. L. Seal* (New Haven, Conn.)

4502:

Wesler, Oscar. A classification problem involving multinomials. Ann. Math. Statist. **30** (1959), 128–133.

Let p_1 and p_2 be two (multinomial) probability k-vectors and ω_1 and ω_2 the corresponding sets of $k!$ vectors obtained by permutation of components. The invariant-Bayes tests are a class which provides a test of any size α of ω_1 against ω_2 which maximizes the minimum power [earlier published references for this result are E. L. Lehmann, same Ann. **21** (1950), 1–26; MR **11**, 528; and J. Kiefer, ibid. **28** (1957), 573–601; MR **19**, 1097]. The author obtains a large sample approximation for the form of the test, which is exact in the binomial case.

J. Kiefer (Ithaca, N.Y.)

4503:

Machek, Josef. On a two-sample procedure for testing Student's hypothesis using mean range. Apl. Mat. **4** (1959), 211–223. (Czech and Russian summaries)

"Stein's two-sample procedure for testing a hypothesis concerning the mean of a normal distribution [Ann. Math. Statist. **16** (1945), 243–258; MR **7**, 213] is modified for the case where the usual mean-square estimate of the standard deviation is replaced by an estimate based on sample range or on mean range of several samples. Expressions for the mean number of necessary observations are derived and some illustrative tables given. Tables of 5% and 10% critical points of the corresponding test statistic are given." (From the author's summary)

I. Olkin (East Lansing, Mich.)

4504:

Deming, W. Edwards; and Glasser, Gerald J. On the problem of matching lists by samples. J. Amer. Statist. Assoc. **54** (1959), 403–415.

Given two or more lists of names which may contain duplicates, random samples are taken from each list and matched to determine the proportion of duplicates in the complete list. This paper presents the statistical theory for this problem, extending work done by Goodman [Ann. Math. Statist. **23** (1952), 632–634; MR **14**, 390]. In particular, the authors give the probability distribution of the number of duplicates in the matched samples as well as various limiting approximations and moments up to the fourth. They indicate by examples how these results can be used to test hypotheses on the level of duplication and also give some results on optimum sample sizes.

D. G. Chapman (Seattle, Wash.)

4505:

Hooper, John W. Simultaneous equations and canonical correlation theory. Econometrica **27** (1959), 245–256.

This paper uses canonical correlation theory in the development of a generalized correlation coefficient for simultaneous equation systems. The canonical correlations are the latent roots of a matrix having the form $(Y'Y)^{-1}Y'X(X'X)^{-1}X'Y = I - D$. The matrix D is equal to $(Y'Y)^{-1}V'V$ where V is the least squares residual matrix. So D may be considered to be a matrix generalization of $1 - R^2$ where R is the coefficient of multiple correlation and $I - D$ may be regarded as a matrix generalization of R^2. Also the latent roots of D are $1 - r^2$, where r is a canonical correlation and the vector of latent roots of $I - D$ may be interpreted as a vector generalization of r^2.

The square of the proposed generalized correlation coefficient, the squared trace correlation $\bar{r}_{yx}{}^2$, is then defined to be $(1/M)$ trace $(I - D)$, where M is the number of variables of Y. When the roles of X and Y are interchanged, the definition is applied to

$$I - D^* = (X'X)^{-1}X'Y(Y'Y)^{-1}Y'X$$

with trace $(I - D^*)$ = trace $(I - D)$. If Λ is the number of variables of X, $\Lambda\bar{r}_{xy}{}^2 = M\bar{r}_{yx}{}^2$. Asymptotic variances of $\bar{r}_{yx}$ and $\bar{r}_{yx}{}^2$ are presented and an application to a simultaneous equation system is given.

P. S. Dwyer (Ann Arbor, Mich.)

4506:

Uematu, Tosio. Note on the numerical computation in the discrimination problem. Ann. Inst. Statist. Math. Tokyo **10** (1959), 131–135.

This note presents a computational method for solving a general discriminant problem with s groups, p characteristics, and n_i individuals in group i. The basic equation takes the form $ANA'x = \lambda Bx$. It is desired to find the eigenvalues λ and the eigenvectors x. The author notes that the basic equation degenerates into an equation with smaller dimension since this analysis may be considered to be a special case of canonical correlation analysis. A technique is presented which requires, as its first step, the solution of only $s - 1$ linear equations. Vectors with only $s - 1$ components are then featured

throughout the solution. A program using a FACOM-128 automatic relay computer is available for any p and s satisfying $p+s \leqq 58$. *P. S. Dwyer* (Ann Arbor, Mich.)

4507:

Wilks, S. S. Recurrence of extreme observations. J. Austral. Math. Soc. **1** (1959/61), part 1, 106–112.

A preliminary set of m independent observations is drawn from a population in which a r.v. x has a continuous but unknown c.d.f. $F(x)$. Let y be the rth from the largest observation in the preliminary sample and let n denote the number of further trials required in order to obtain k observations which exceed y. The p.d. of n is found to be

$$p(n) = \frac{1}{n}\,\frac{B(k+r,\, m+n-k-r+1)}{B(k,\, n-k+1)B(r,\, m-r+1)} \quad (n = k, k+1, \cdots).$$

The asymptotic distribution of n/m as $m \to \infty$ is obtained and has a simple form independent of $F(x)$.

I. Olkin (East Lansing, Mich.)

4508:

Walsh, John E. Large sample nonparametric rejection of outlying observations. Ann. Inst. Statist. Math. Tokyo. **10** (1959), 223–232.

"Let $x_1 \leqq, \cdots, \leqq x_n$ be independent observations from continuous populations. The null hypothesis, H_0, is that these observations are a sample. The alternative hypothesis is that the i smallest observations are too small (or that the i largest observations are too large) to be consistent with H_0. Here i is a small number and should be specified without knowledge of the observation values. The common population hypothesized for the null case is assumed to be well-behaved but no specific assumptions are made about its shape. The alternative that the i smallest observations are too small is accepted if a statistic of the form $x_i-(1+A)x_{i+1}+Ax_k$ is negative, where $A>0$, k is the largest integer contained in $i+\sqrt{2n}$, and n is sufficiently large. Similarly, the alternative that the i largest observations are too large is accepted if $x_{n+1-i}-(1+A)x_{n-i}+Ax_{n+1-k}$ is positive. Two-sided tests are obtained as combinations of these one-sided tests. For A suitably chosen, an approximate upper bound for the significance level of a test is evaluated from Tchebycheff's inequality." (Author's summary)

F. J. Anscombe (Princeton, N.J.)

4509:

Wallace, David L. Simplified beta-approximations to the Kruskal-Wallis H test. J. Amer. Statist. Assoc. **54** (1959), 225–230.

The null distribution of B_2, where $(N-1)B_2=H$ is the Kruskal-Wallis statistic for one-way analysis of variance of ranks [same J. **47** (1952), 583–621; **48** (1953), 907–911; MR **14**, 391] is approximated by a Beta-distribution by equating means and variances. It is indicated that the resulting test is better in the extreme tails than the Beta-approximation of Kruskal and Wallis, which requires the calculation of B_1, the ratio of H to the maximum possible value of H. This test, a seemingly more desirable simplification for the case of nearly equal sample sizes, the equivalent of the ordinary F-test, and the exact test are studied for three small sample cases. Bounds are given on the difference between significance levels of the former three tests. *J. H. Stapleton* (East Lansing, Mich.)

4510:

Menger, Karl. Optimal differences in computing probable derivatives. J. Math. Pures Appl. (9) **38** (1959), 245–252.

Suppose that a one parameter family of independent distributions be given; let the mean be $f(x)$ and the standard deviation be $g(x)$. Each point P of abscissa x_0 can be joined to each point P of abscissa $x \neq x_0$ by a straight line. The author investigates the optimal choice of x so that $P_k(x)$ be a maximum, where $P_k(x)$ is the probability that, for a given $k>0$, the slope of P_0P differ from $f'(x_0)$ by at most k; he discusses the cases of a family of uniform distributions and of a family of normal distributions.

R. G. Stanton (Waterloo, Ont.)

4511:

Sibuya, Masaaki; and Haga, Toshiro. Orthogonal polynomials without constant term. Ann. Inst. Statist. Math. Tokyo. **10** (1959), 209–222.

Define $X_n^{(N)}(x)$ $(1 \leqq n \leqq N)$ as polynomials in x of degree n, with no constant term, which are orthogonal with respect to summation at N equidistant points (here we simplify $X_n^{(N)}$ to X_n):

$$X_n(0) = 0; \quad \sum_{x=1}^{N} X_n(x)X_m(x) = 0 \quad \text{for } n \neq m.$$

These conditions define $X_n(x)$ except for multipliers, which are used to reduce the X's to smallest possible integers. These polynomial values can be used to simplify the estimation of parameters in the linear model:

$$y(x) = \sum_{n=1}^{r} \beta_n{}^*x + \varepsilon_x = \sum_{n=1}^{r} \beta_n X_n(x) + \varepsilon_x,$$

where $r \leqq N$. The estimators are $\beta_n = B_n/S_n$, where $B_n = \sum_{x=1}^{N} y(x)X_n(x)$ and $S_n = \sum_{x=1}^{N} [X_n(x)]^2$. Tables of $X_n(x)$, multipliers and S_n are presented for $n=1(1)4$ and $N=2(1)23$; of course, n only goes to N for $N=2, 3$. A recurrence formula is given to derive $X_n(x)$ from $X_{n-1}(x)$ and $X_{n-2}(x)$. *R. L. Anderson* (Raleigh, N.C.)

4512:

★Acton, Forman S. Analysis of straight-line data. John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London; 1959. xiii + 267 pp. $9.00.

This book encompasses some techniques for fitting straight lines to data. The author states in his preface, "I did not intend this book for classroom use". He then goes on to state that the book is "for home perusal by the statistically self-educating physical scientist and engineer". It is not clear to the reviewer what level of self-education the author had in mind but it is clear that this level changed many times in the text. He has accepted a serious responsibility to be lucid, precise, and above all, consistent if he recommends that the book can be read without the support of a classroom environment. The author could have exercised this responsibility with much greater care, for there are parts of the text where he does show insight and a lucid and pleasant style of writing. However, too often the reader is required to have knowledge of statistical theory beyond the level of the material

being presented. Also, while the author criticizes others for presenting material in a very mechanical way he has not avoided this pitfall, especially in Chapters 3, 6, and 7.

It is surprising that no exercises or guides are presented to assist the reader in evaluating his progress. The need for additional examples is apparent if one notes that only ten sets of data appear in the text, nine of which might be said to belong to the area of chemistry. Considering the wealth of physical science and engineering data, it is surprising to find the bivariate normal distribution being illustrated in the text by a set of data on the dermal ridge counts for the little fingers of both hands in pairs of parents and children.

Except for brief mention of possible computational advantages that may result from having equally spaced observations, and remarks such as the plea, "Take time to design your experiment to answer the important questions", the book fails to illustrate the importance of "where" or "how" data should be obtained. This is a serious weakness in the book. From the reviewer's experience he has found physical scientists and engineers in great need of understanding how the selection of data (design of the experiment) influences the analysis and possible inferences they might wish to make.

Non-parametric and quick and easy numerical procedures are reviewed in the text. It is nice to see this type of material and it will be of value for certain areas of application. However, a careful discussion of when such techniques should be considered is needed. Certainly if the experiment is expensive it behoves the experimenter to use the "best" available statistical techniques. With the present availability of electronic computers one now can usually afford to use more elaborate analytical techniques.

In a few places the notation may cause the reader unnecessary difficulty, for example, y may denote an observed data point yet, for instance on page 17, it also denotes a model value. In Chapter nine it would help the reader to mention that the observations are assumed to be ordered according to their magnitude.

Finally, the preface could easily give the impression that the "classical" least squares approach is about twenty years old; this is unfortunate. Neither the text nor the list of references adequately indicates the depth or beauty of the subject. For a book of this size the reviewer would like to have seen some mention of the history of the subject or at least a few relevent references, yet no mention is made of the development of the subject, not even the basic contribution of Gauss. Also, it would seem to the reviewer that the reader is entitled to a survey or listing of the literature of the past 20 years related to the analysis of "straight-line data" when both variables are subject to error.

The material by chapter is as follows: Chapter 1—"The Choice of a Model"—A brief expository discussion of several types of "linear" models. It is unfortunate that the author neglected to mention the fact that the statistical estimation of the coefficients of a model which is polynomial in the independent variable (which is measured without error) is still a linear problem in so far as the type of statistical estimation procedures that are used. Here or in chapter seven (the fitting of polynomials) would have been an ideal place to have mentioned this fact. Chapter 2—"The Classical Model: x known without error; variance of y constant" consists of the following four sections: Section 1—"Classical Fitting Methods": Contains a least squares approach to the estimation of the parameters without indicating why this procedure has merit, for example, its optimum minimum variance properties. Section 2—"The Setting of Confidence Limits": Part of the presentation on confidence regions is done well. However, much of the material in this section cannot be appreciated without some familiarity with the bivariate normal distribution, yet this topic is not considered until Chapter 4. The value of this section could be considerably improved if the text contained an explanation for the concept, degree of freedom. Also, diagrams 10 and 11 would be more meaningful if they displayed the mean squares instead of the sums of squares. It would also be helpful if the text contained a discussion and an example to illustrate the intrinsic difference between interpolation and extrapolation, especially with regard to the establishment of confidence regions. The section closes with a presentation of population tolerance limits, however, from a pedagogical point of view this material might form a separate section. Also, the reader should be informed that some literature and some criteria exist for deciding whether confidence regions should be symmetrical or asymmetrical, connected or disconnected. Section 3—"Modified Models for one Dependent Variable": Here one finds in a short space of 25 pages a brief survey of some techniques which do not depend upon, or which replace normal distribution assumptions. Section 4—"Two Lines or One": Two different ideas are considered here, one concerns the comparison of two different methods of measurement of the same data, the other concerns the comparison of two different sets of data. The reader who is not familiar with the techniques of the Analysis of Variance may have difficulty here. Chapter 3—"Regression with Several Values of y for Each Known x":—Again, unless the reader is knowledgeable in the technique of the Analysis of Variance he may find this chapter difficult to understand. The last part of the chapter, for all intensive purposes, is on a different subject—namely, "Supplement on Finite Populations". Here various means and variances are derived for finite populations, including a case where the error in y is dependent on the value of x. Chapter 4—"Samples from Bivariate Normal Populations". Most of the subject matter in this chapter has been presented in a better manner by other writers. The subject of this chapter ought to come earlier in the text. Chapter 5—"Regression with Both x and y in Error". Many specifications of the problem concerning the linear structural relationship between two variables subject to errors in both variables can be found in the literature [see for example J. Wolfowitz, Skand. Aktuarietidskr. **35** (1952), 132–151; MR **14**, 776]. The presentation here is essentially restricted to using maximum likelihood estimators where it is assumed that the errors have a bivariate normal distribution and that 2 of the 3 terms of the covariance matrix are known. Chapter 6—"Several Lines; the Analysis of Variance": A weak presentation of the arithmetic of a two-way analysis of variance. This is followed by a lucid discussion of the pitfalls that can occur from performing repeated tests on the same data. The presentation of (Tukey's) multiple-comparison tests is much too brief. Chapter 7—"The Exposure of Curvature: Orthogonal Polynomials". Chapters 8, 9, and 10 contain brief expository consideration of the following important topics: "The Use of Transformations", "The Rejection of Unwanted Data", and "Cumulative Data; The Fading Line".

The following pages have been found to contain errors: 17, 27, 99, 133, 185, and 195.

M. E. Muller (New York, N.Y.)

4513:

Masuyama, Motosaburo. On the optimality of balanced incomplete block designs. Rep. Statist. Appl. Res. Un. Jap. Sci. Engrs. **5** (1957/58), 4–8.

Among all incomplete block designs with given parameter v, b, k the balanced designs minimize the maximum of the variances of all contrasts.

H. B. Mann (Columbus, Ohio)

4514:

★Elfving, G. Minimax character of balanced experimental designs. Treizième congrès des mathématiciens scandinaves, tenu à Helsinki 18–23 août 1957, pp. 69–76. Mercators Tryckeri, Helsinki, 1958. 209 pp. (1 plate)

Suppose that a balanced incomplete block design with parameters v, b, k exists. The author shows that this balanced design is the only one among all incomplete block designs which minimizes the maximum of the variances of all standard contrasts. The same result has been obtained by M. Masuyama [see preceding review].

H. B. Mann (Columbus, Ohio)

4515:

Bechhofer, Robert E.; Elmaghraby, Salah; and Morse, Norman. A single-sample multiple-decision procedure for selecting the multinomial event which has the highest probability. Ann. Math. Statist. **30** (1959), 102–119.

The problem of selecting the multinomial event which has the highest probability is formulated as a multiple decision problem. "The probability of a correct selection is to be $\geq P^*$ whenever the true (but unknown) ratio of the largest to the second largest population probabilities $\geq \theta^*$." A single sample procedure which meets the requirements is proposed. Tables and sample calculations are provided. (From the authors' summary)

H. Teicher (Lafayette, Ind.)

4516:

Kesten, Harry; and Morse, Norman. A property of the multinomial distribution. Ann. Math. Statist. **30** (1959), 120–127.

For a sample of $N \geq 1$ observations from a multinomial population of $K \geq 2$ classes it is proved that among all parameter configurations with $p_{[k]} \geq \theta^* p_{[k-1]}$ ($\theta^* > 1$ and $p_{[1]} \leq p_{[2]} \leq \cdots \leq p_{[k]}$ are the ordered "success probabilities") the probability of a correct selection (according to the plan of the preceding paper) is minimized when

$$(\theta^*)^{-1} p_{[k]} = (\theta^* + k - 1)^{-1} = p_{[i]} \quad (i = 1, 2 \cdots, k-1).$$

(This justifies the procedure of the prior paper [#4515].)

H. Teicher (Lafayette, Ind.)

4517:

Basu, D. On sampling with and without replacement. Sankhyā **20** (1958), 287–294.

It is known that an average of n sample observations obtained without replacement from a finite population of N real numbers is an unbiased estimate of the population mean; its variance is $(N-n)/(N-1)$ times that of the variance of the average of n observations obtained with replacement. Hence, sampling without replacement is often considered "more efficient" than sampling with replacement. However, the author points out that if the experimenter pays only for non-repeated population elements observed, then the situation is reversed. That is, in sampling with replacement the expected number of non-repeated observations is $C = N(1-((N-1)/N)^n)$; and a sample average based on $[C]$ observations drawn without replacement has variance greater than σ^2/n. On the other hand, it is proved that in sampling with replacement, if instead of averaging the n observations, one averages the random number of non-repeated observations, one obtains an unbiased estimate whose variance is less than σ^2/n. There is some discussion of the role of sufficiency of sampling until a fixed number of non-repeated observations are obtained, and of sampling with unequal probabilities.

M. Dwass (Evanston, Ill.)

4518:

Dorogovcev, A. Ya. Statistical analysis of a stochastic difference equation. Dopovidi Akad. Nauk Ukraïn. RSR **1959**, 120–124. (Ukrainian. Russian and English summaries)

"Let x_t be a stochastic process depending on an integral-valued parameter t and suppose that the following difference equation is verified

$$x_t + a_1 x_{t-1} + \cdots + a_p x_{t-p} + a_0 = \sum_{j=0}^{\nu} b_j \xi_{t-j},$$

where a_i $(0 \leq i \leq p)$ and b_k $(0 \leq k \leq \nu)$ are unknown parameters and ξ_t independent normalized random variables (i.e., $E(\xi_t) = 0$, $E(\xi_t^2) = 1$). By making use of a sequence of observations $x_1, \cdots, x_N$ of the process the parameters a_j and b_k are estimated. Further, biased estimates are discussed and estimates for the a_j-parameters are given; these estimates are consistent and their joint distribution function becomes asymptotically Gaussian, as $N \to \infty$." (From the author's summaries)

R. Theodorescu (Bucharest)

4519:

Stoller, D. S.; and Stoller, L. C. Calculating the coefficients of certain linear predictors. Math. Tables Aids Comput. **13** (1959), 122–124.

With $n+1$ equally spaced observations $x_j, j = 0, 1, \cdots, n$, we wish to find the linear predictor $y_{n+1} = \sum_{j=0}^{n} a_j x_j$ of the next observation, x_{n+1} (not yet made) under the assumptions $\sigma_{x_j} = \sigma$, $E(x_j)$ is a kth degree polynomial in j, and $E(y_{n+1}) = E(x_{n+1})$ for an arbitrary polynomial of degree k. The authors obtain a formula featuring orthogonal polynomials for the values of a_j which minimize $\sigma^2_{y_{n+1}}/\sigma^2$. The use of certain polynomials, orthogonal over any set of points symmetric about the origin, is indicated.

P. S. Dwyer (Ann Arbor, Mich.)

NUMERICAL METHODS

See also 4164, 4228, 4512, 4519, 4705, 4833.

4520:

Shanks, Daniel. A sieve method for factoring numbers of the form n^2+1. Math. Tables Aids Comput. **13** (1959), 78–86.

The method of the title has been applied on an IBM704

to factorize n^2+1, for all $n \leqq 180000$. Some of the results about the density of primes and reducible numbers [cf. J. Todd, Amer. Math. Monthly **56** (1949), 517–528; MR **11**, 159] are summarized in a table. There is a discussion of the results and their relations to various conjectures. The method has also been applied to study numbers of the form n^2+a, $a=\pm 2, \pm 3$. *John Todd* (Pasadena, Calif.)

4521:
Muller, Mervin E. A comparison of methods for generating normal deviates on digital computers. J. Assoc. Comput. Mach. **6** (1959), 376–383.

A summary of six methods for the generation of normally distributed random variables from a sequence of uniformly distributed random numbers. The author has programmed each method for the IBM704 and presents a table comparing their speeds and storage requirements.
B. A. Chartres (Sydney)

4522:
Gautschi, Walter. Note on bivariate linear interpolation for analytic functions. Math. Tables Aids Comput. **13** (1959), 91–96.

This is an investigation of the variation of the error-bound in the interpolation in the title, in both the Cartesian and polar cases. It is shown how some improvement in the usual methods can be obtained.
John Todd (Pasadena, Calif.)

4523:
★Arend, S. Quelques aspects du problème de la représentation de phénomènes expérimentaux. Actes du colloque de calcul numérique, Périgueux, 1957, pp. 45–54. Publ. Sci. Tech. Ministère de l'Air, Notes Tech. no. 80, Paris, 1959. vii+87 pp. 1800 francs.

Revue de divers procédés pour la détermination rapide des coefficients d'une représentation approchée.
J. Kuntzmann (Grenoble)

4524:
★Juricic, Henri. Sur l'application de la méthode des moindres carrés au lissage des courbes expérimentales. Actes du colloque de calcul numérique, Périgueux, 1957, pp. 55–82. Publ. Sci. Tech. Ministère de l'Air, Notes Tech. no. 80, Paris, 1959. vii+87 pp. 1800 francs.

Exposé avec discussion et exemples de l'ajustement d'une courbe par moindres carrés.
J. Kuntzmann (Grenoble)

4525:
Salton, Gerard. The use of the central limit theorem for interpolating in tables of probability distribution functions. Math. Tables Aids Comput. **13** (1959), 213–216.

4526:
Murnaghan, F. D.; and Wrench, J. W., Jr. The determination of the Chebyshev approximating polynomial for a differentiable function. Math. Tables Aids Comput. **13** (1959), 185–193.

In a previous paper [David Taylor Model Basin Report 1175, 1958] the authors gave a method for the approximation of a differentiable function by polynomials. In this paper they give a method for the approximation of a differentiable function $f(x)$ by a "Chebyshev approximating polynomial", as follows: The "entering" polynomial approximation is

$$P_n^{(0)}(x) = \tfrac{1}{2}a_0+a_1T_1(x)+a_2T_2(x)+\cdots+a_nT_n(x),$$

which is the truncated Chebyshev polynomial expansion of $f(x)$. Then by an iterative process for which formulas are given, polynomials $P_n^{(k)}(x)$ $(k=1, 2, \cdots)$ are determined which approximate successively closer to $P_n^*(x)$, the Chebyshev approximation, of maximum degree n, to $f(x)$ over $-1 \leqq x \leqq 1$. This polynomial is such that the maximum of $|f(x)-P_n^*(x)|$ over the interval is less than the maximum of $|f(x)-P_n(x)|$ over $-1 \leqq x \leqq 1$, where $P_n(x)$ is any other polynomial of degree not exceeding n. Numerous examples that illustrate the method are given.
E. Frank (Chicago, Ill.)

4527:
Curtis, Philip C., Jr.; and Frank, Werner L. An algorithm for the determination of the polynomial of best minimax approximation to a function defined on a finite point set. J. Assoc. Comput. Mach. **6** (1959), 395–404.

The algorithm is a variant of the method of equalization of maxima [cf. second algorithm in E. Ya. Remez, *Obščie vyčislitel'nye metody Čebyševskogo približeniya*, Izdat. Akad. Nauk Ukrain. SSR, Kiev, 1957; MR **19**, 580; also E. P. Novodvorskiĭ and I. Š. Pinsker, Uspehi Mat. Nauk (N.S.) **6** (1951), no. 6 (46), 174–181; MR **13**, 728]. There are several remarks on practical details and four numerical examples. {In formula (4) the summation sign should be replaced by a product sign.}
Walter Gautschi (Oak Ridge, Tenn.)

4528:
Corbató, Fernando J.; and Uretsky, Jack L. Generation of spherical Bessel functions in digital computers. J. Assoc. Comput. Mach. **6** (1959), 366–375.

A method that is becoming increasingly popular for the generation of Bessel functions $J_n(x)$ on an automatic computer is based on the use of the recurrence relation

$$\mathscr{C}_{n+1}(x)-\frac{2n}{x}\mathscr{C}_n(x)+\mathscr{C}_{n-1}(x) = 0$$

which has a general solution

$$\mathscr{C}_n(x) = AJ_n(x)+BY_n(x).$$

If we start with two values $J_0(x), J_1(x)$ for given x, then theoretically $J_2(x), J_3(x), \cdots, J_n(x)$ can be generated in turn. In practice, rounding errors at every stage gradually modify the constants A, B to, say, A^*, B^*. So, although we start with $A=1$, $B=0$, we shall find $B^* \neq 0$. This does not matter so long as $n<x$, because J_n and Y_n are both oscillatory with n and have equal-sized and interlacing oscillations, so that B^* will remain small relative to A, while $A^*=A=1$ approximately. As soon as x exceeds n, however, $J_n \to 0$, $|Y_n| \to \infty$ fairly rapidly, and B^*Y_n introduces serious, and eventually overwhelming error.

On the other hand, if initially $A=0$, $B=1$, there is no trouble from A^* as J_n remains at first of size comparable to that of Y_n and then, when $n>x$, rapidly evaporates relative to Y_n.

We can, however, obtain $J_n(x)$ by working with decreasing n, starting from $n=N$, well into the region $n>x$, and somewhat beyond the last value of n needed in our final

results. We can start with $\mathscr{C}_{N+1}=0$, $\mathscr{C}_N=1$, to which corresponds

$$\mathscr{C}_N(x) = AJ_N(x)+BY_N(x)$$

for definite values of A, B. Now as n diminishes to $n=x$, $Y_n(x)$ diminishes also, while $J_n(x)$ decreases. The term $B^*Y_n(x)$ thus becomes negligible, while $A^*J_n(x)$, and hence $\mathscr{C}_n(x)$, becomes proportional to $J_n(x)$, to the precision to which we work (supposed a constant number of figures). We have only to determine A^* and to multiply by its reciprocal to recover $J_n(x)$ to the desired precision.

We may do this either by comparing $AJ_0(x)$ with the known value of $J_0(x)$; or we may use the series relation

$$1 = J_0(x)+2\sum_1^\infty J_{2r}(x)$$

to determine A^*.

The paper here reviewed discusses the generation of the spherical Bessel functions $j_n(x)$, $i_n(x)$, $k_n(x)$, where $j_n(x)=\sqrt{(2/\pi x)}J_{n+1/2}(x)$, etc. by similar procedures.

J. C. P. Miller (Madison, Wis.)

4529:

Meixner, J.; and Wells, C. P. Improving the convergence in an expansion of spheroidal wave functions. Quart. Appl. Math. **17** (1959), 263–269.

In order to facilitate the numerical computation of certain infinite series involving spheroidal wave functions, the authors first replace, in lator terms of the series, spheroidal wave functions by their asymptotic forms involving Legendre functions, and then use asymptotic forms for some of the Legendre functions to obtain series of which at least a part can be summed explicitly. Some numerical studies are given which seem to indicate that in certain ranges of the parameters involved two- or three-place accuracy can be obtained by thus approximating all but the first six terms of the series. Many more terms would be needed if the same accuracy was to be achieved without the use of the comparison series.

A. Erdélyi (Pasadena, Calif.)

4530:

Goldstein, M.; and Thaler, R. M. Recurrence techniques for the calculation of Bessel functions. Math. Tables Aids Comput. **13** (1959), 102–108.

The recursion formula $F_{\nu-1}(x)+F_{\nu+1}(x)=2\nu x^{-1}F_\nu(x)$ is to be used going downward from the initial values $F_{\nu+M+1}(x)=0$, $F_{\nu+M}(x)=a$, where a is any convenient constant, as a means of computing $J_{\nu+n}(x)$, $n\leq N$, $0\leq\nu<1$. $F_{\nu+n}(x)=\alpha J_{\nu+n}(x)+\beta Y_{\nu+n}(x)$, and M can be taken so large that $F_{\nu+n}(x)\cong\alpha J_{\nu+n}(x)$ for $n\leq N$. With the $F_{\nu+n}(x)$ determined from the recursion relation, α can be found from a truncated Neumann series for $x^{\nu}2^{-\nu}$, and thus $J_{\nu+n}(x)$ obtained. The function $Y_{\nu+n}(x)$ can then be found with the aid of the Wronskian relation. A similar scheme handles the modified Bessel functions $I_{\nu+n}(x)$ and $K_{\nu+n}(x)$.

R. G. Langebartel (Urbana, Ill.)

4531:

Uhlmann, Werner. Eine wahrscheinlichkeitstheoretische Begründung der Integrationsformeln von Newton-Cotes. Z. Angew. Math. Phys. **10** (1959), 189–207. (English summary)

Following Blanc [Comment. Math. Helv. **26** (1952), 225–241; MR **14**, 691], Blanc and Liniger [Z. Angew. Math. Mech. **35** (1955), 121–130; MR **16**, 1154], and Morgenstern [Math. Nachr. **13** (1955), 161–164; MR **17**, 791], the author regards an integrand $x(t)$ as a realization of a stochastic process $\xi(t)$, and selects quadrature formulas to minimize the expected value of the squared-error over the ensemble.

For fixed abscissas, the Newton-Cotes weights are shown to be asymptotically best for any smooth covariance function, as the interval of integration shrinks to a point. Details are as follows.

Let $E[\xi(t)]=0$, $E[\xi^2(t)]=1$, $E[\xi(t)\xi'(t)]=R(t-t')$. Let $S(t)=\int_0^t R(\tau)d\tau$. Let

$$(*)\qquad \eta = \int_{-nh}^{nh}\xi(t)dt-\sum_{i=-n}^{n}a_i\xi(ih).$$

Methods for choosing $\{a_i\}$ to minimize $E(\eta^2)$ are given in terms of R and of S. The author also treats related formulas with an even number of ordinates and weights a_i.

Assume that $R(t)$ is in class C^{4n-5}, and that a certain determinant is nonzero. Now $E(\eta^2)$, and hence also the minimizing a_i (which may not be unique), depend only on h. If one chooses these $a_i(h)$ to be sufficiently smooth, the author proves the startling result that the leading terms of the Taylor expansions of the minimizing $a_i(h)$ are the Newton-Cotes coefficients corresponding to the $2n+1$ abscissas of (*). If $R(t)$ is in class C^{4n+8}, and if the a_i in (*) are the Newton-Cotes coefficients, it is proved that $E(\eta^2)$ is $O(h^{4n+6})$, as $h\to0$. Moreover, if $R(t)$ is in class C^{4n+7}, and if the $a_i(h)$ minimize $E(\eta^2)$ in (*), then $E(\eta^2)$ is $O(h^{4n+6})$ but not $O(h^{4n+7})$. Thus the Newton-Cotes integration formulas receive a new justification, even though they do not quite minimize $E(\eta^2)$ for finite h. There are other results.

G. E. Forsythe (Stanford, Calif.)

4532:

Goodey, W. J. Solution of modified linear simultaneous equations. An investigation associated with the analysis of redundant structures. Aircraft Engrg. **31** (1959), 358–359, 364.

4533:

Rutishauser, Heinz. Zur Matrizeninversion nach Gauss-Jordan. Z. Angew. Math. Phys. **10** (1959), 281–291. (English summary)

This paper presents a somewhat different approach to the well-known Gauss-Jordan procedure for inverting matrices. Let $(x_1, \cdots, x_n)$ be any point in R^n and $(y_1, \cdots, y_n)$ its image under the given linear transformation A of R^n. The linear operator $A_{pq}^{(1)}$ which maps $(x_1, \cdots, x_{q-1}, y_p, x_{q+1}, x_n)$ into $(y_1, \cdots, y_{p-1}, x_q, y_{p+1}, \cdots, y_n)$ is well defined provided the pivotal element a_{pq} in the matrix of A is different from zero. Writing $A_{pq}^{(1)}=J_{pq}A$ one verifies that the repeated application of such transformations J_{pq} is equivalent to the Gauss-Jordan procedure (provided, of course, that all p, q are chosen suitably). No geometrical interpretation of these transformations is given.

W. C. Rheinboldt (Syracuse, N.Y.)

4534:

Schechter, Samuel. On the inversion of certain matrices. Math. Tables Aids Comput. **13** (1959), 73–77.

Lagrangian interpolation polynomials are used to obtain

simple formulas for the elements of the inverse and related quantities of the matrix $H=\{h_{ij}\}$, where $h_{ij}=(a_i-b_j)^{-1}$ $(1\leq i,j\leq n)$. *I. R. Savage* (Minneapolis, Minn.)

4535:

Maruki, Goichi. Solving inhomogeneous linear equations. Bull. Fukuoka Gakugei Univ. III **8** (1958), 13–15.

The author describes an iterative procedure for solving a set of linear algebraic equations $My+b=0$, where M is a square matrix of order n and where b is a given column matrix. It is assumed that the eigenvalues are known and that they are distinct. The method involves performing 3 sets of n iterations using the method known as the method of simultaneous displacements, or the Jacobi method. A suitable linear combination is obtained of each set of iterants. By the use of the Aitken extrapolation procedure as applied to the three vectors so determined, one obtains the exact solution. A numerical example is given to illustrate the method and to show that it applies even when M is singular, as long as a solution does exist. However, the author's statement that the exact solution is obtained is misleading since the solution is clearly not unique. Also, it should be pointed out that if M is nonsingular, then by the use of the method of L. F. Richardson [Philos. Trans. Roy. Soc. London Ser. A **210** (1910), 307–357] one can obtain the solution in n iterations provided that the eigenvalues of M are known.

D. M. Young, Jr. (Austin, Tex.)

4536:

Cheney, Ward; and Goldstein, Allen A. Note on a paper by Zuhovickiĭ concerning the Tchebycheff problem for linear equations. J. Soc. Indust. Appl. Math. **6** (1958), 233–239.

The paper elaborates a paper by Zuhovickiĭ, [Dokl. Akad. Nauk SSSR **79** (1951), 561–564; MR **13**, 285]. It is concerned with the Tchebycheff problem for linear equations. The authors give a detailed justification of the method described in the earlier paper and suggest alternatives. *M. R. Hestenes* (Los Angeles, Calif.)

4537:

Ansorge, R. Das Hertwigsche Iterationsverfahren zur Auflösung linearer Gleichungssysteme als Gesamt- und Einzelschrittverfahren. Z. Angew. Math. Mech. **39** (1959), 248–249.

4538:

★Aparo, Enzo. Sulle equazioni algebriche matriciali. Symposium on modern computers: Proceedings of the Rome Symposium (17–18 October 1956) organized by the Preparatory Committee of the International Computation Center under the auspices of Unesco, pp. 27–29. Libreria Eredi Virgilio Veschi, Rome. vii + 94 pp.

4539:

Bertram, G. Fehlerabschätzung für das Ritz-Galerkinsche Verfahren bei Eigenwertproblemen. Z. Angew. Math. Mech. **39** (1959), 236–246. (English, French and Russian summaries)

Part I of this paper [same Z. **37** (1957), 191–201; MR **19**, 462] gave lower bounds for eigenvalue approximations got by the Ritz-Galerkin method. Here the results are applied and improved for special classes of ordinary eigenvalue problems. Some numerical examples are given.

R. C. T. Smith (Armidale)

4540:

Lotkin, Mark. Determination of characteristic values. Quart. Appl. Math. **17** (1959), 237–244.

In a previous paper [same Quart. **14** (1956), 267–275; MR **19**, 832] the author presented a numerical method for reducing an arbitrary complex matrix to triangular form by a sequence of unitary transformations. In the present paper, a refinement of the method is given, which, for a considerable number of certain matrices, leads to more rapid convergence. *B. N. Moyls* (Vancouver, B.C.)

4541a:

Ostrowski, A. M. On the convergence of the Rayleigh quotient iteration for the computation of the characteristic roots and vectors. III. (Generalized Rayleigh quotient characteristic roots with linear elementary divisors). Arch. Rational Mech. Anal. **3**, 325–340 (1959).

4541b:

Ostrowski, A. M. On the convergence of the Rayleigh quotient iteration for the computation of the characteristic roots and vectors. IV. (Generalized Rayleigh quotient for nonlinear elementary divisors). Arch. Rational Mech. Anal. **3**, 341–347 (1959).

[For parts I and II see same Arch. **1** (1958), 233–241; **2** (1958/59), 423–428; MR **21** #427.] The present papers are the third and the fourth of a sequence of papers on the convergence of Rayleigh quotient iterations for computation of characteristic roots of matrices. In paper number III it is shown that under certain rather general situations the convergence of successive estimates of the eigenvalue is cubic in character. The paper is written in such a way as to exhibit the character of the convergence in all cases. In paper number IV a quadratically convergent iteration rule is given for computing an eigenvalue of a matrix to which corresponds a non-linear elementary divisor. In each of these papers illustrative numerical examples are given verifying the theory.

M. R. Hestenes (Los Angeles, Calif.)

4542:

★Salles, Francisque. Lieu des racines d'une équation algébrique dépendant d'un paramètre. Application à la stabilité et au guidage des fusées. Préface de L. Sackmann. Publ. Sci. Tech. Ministère de l'Air, no. 351, Paris, 1959. x + 69 pp. 1810 francs.

Étude élémentaire du lieu dans le plan complexe des racines de l'équation $n(p)+Kd(p)=0$, K paramètre. Exemple de la stabilité d'une fusée en vol vertical.

J. Kuntzmann (Grenoble)

4543:

Lance, G. N. Solution of algebraic and transcendental equations on an automatic digital computer. J. Assoc. Comput. Mach. **6** (1959), 97–101.

The author considers the problem of finding the zeros

of $f(z)=u+iv$ in the complex plane. This is done by minimixing the function $S=|u|+|v|$ by iteration of the form

$$x_{i+1} = x_i+S(x_i, y_i)-S(x_i+h, y_i),$$
$$y_{i+1} = y_i+S(x_i, y_i)-S(x_i, y_i+h).$$

This method is a modification of the method of steepest descent. A comparison of this method is made with the method of steepest descent and with the method of alternate adjustment of variables. Numerical examples are given. *M. R. Hestenes* (Los Angeles, Calif.)

4544:

Popov, B. S. Sur un procédé de résolution numérique des équations. Bull. Sci. Math. (2) **81** (1957), 29–31.

Let f be a polynomial of degree n in a real variable x. The author sketches two derivations of an apparently new formula for determining a very accurate approximation $\bar{\alpha}_1$ to a zero α_1 of f, starting from a fair approximation $\bar{x}$:

$$\bar{\alpha}_1 = \bar{x}-\Phi(\bar{x})\frac{L_{k-1}(\bar{x})}{L_k(\bar{x})}.$$

Here $\Phi(x)=f(x)/f'(x)$, $L_0(x)=1$, and

$$L_k(x) = \sum_{i=1}^{k}\frac{(-1)^{i-1}}{i!}\Phi^{i-1}(x)\Phi^{(i)}(x)L_{k-i}(x).$$

A bound is given for the error $|\bar{\alpha}_1-\alpha_1|$, which is $O(|\bar{x}-\alpha_1|^{k+2})$, as $\bar{x}\to\alpha_1$ {probably only when α_1 is a simple zero}. *G. E. Forsythe* (Stanford, Calif.)

4545:

Ralston, A. A family of quadrature formulas which achieve high accuracy in composite rules. J. Assoc. Comput. Mach. **6** (1959), 384–394.

The author proposes a quadrature formula in which the integration over a subinterval $[X_0, X_0+rh]$ has the form

$$\int_{X_0}^{X_0+rh} f(X)dX = H_0[f(X_0+rh)-f(X_0)]+\sum_{j=1}^{r}H_jf(a_j)+E_r,$$

where $E_r=Kh^{2r+2}f^{(2r+1)}(n_1)$. Since H_0 appears alternately with factors ± 1, the terms involving H_0 cancel except for the end intervals. Formulas are given to establish the values of H_j and a_j. The author suggests use of $r=1$ or 2 and gives explicit values of H_j and a_j, for these cases. The method is suitable for analytically given integrals since the a_j are irrational numbers, in general.

P. C. Hammer (Madison, Wis.)

4546:

Quade, W. Über die Stabilität numerischer Methoden zur Integration gewöhnlicher Differentialgleichungen erster Ordnung. Z. Angew. Math. Mech. **39** (1959), 117–134. (English, French and Russian summaries)

A numerical method which approximates the solution of the initial value problem $y'=f(x, y)$, $y(x_0)=y_0$ at equidistant points $\{x_\nu\}$ by the solution $\{y_\nu\}$ of a difference equation is called stable if $\bar{y}_\nu-y_\nu\to 0$ as $\nu\to\infty$ for any other solution $\{\bar{y}_\nu\}$ of the same difference equation. Otherwise, the method is called unstable. The author examines three well-known methods for stability in this sense and suggests modifications which may turn an unstable method into a stable one. *Walter Gautschi* (Oak Ridge, Tenn.)

4547:

Kuntzmann, J. Évaluation de l'erreur sur un pas dans les méthodes à pas séparés. Chiffres **2** (1959), 97–102. (English, German and Russian summaries)

The author produces a sequence of formulas expressing the error in a single step, using any of the Runge-Kutta type integration procedures for second-order differential equations, in terms of the values of the numerical solution at the present, preceding, and following points. The analysis is based on certain relations between the values of an arbitrary function and its derivatives at equally-spaced points. Formulas with errors of increasing order, up to 7, in the step-size h, are given. He suggests that this technique is more useful than those formulas involving higher partial derivatives of the functions on the right-hand side, and states that the various formulas have proved useful in practice.

J. W. Carr, III (Chapel Hill, N.C.)

4548:

Conte, S. D.; and Dames, R. T. An alternating direction method for solving the biharmonic equation. Math. Tables Aids Comput. **12** (1958), 198–205.

The authors extend to biharmonic difference equations the implicit alternating-direction solution method developed for Poisson's equation by J. Douglas and H. Rachford [Trans. Amer. Math. Soc. **82** (1956), 421–439; MR **18**, 827]. They consider an idealized square clamped plate, and introduce a square grid and the usual finite-difference equations. The algorithm for solving the difference equations is implicit by lines, alternating vertical lines with horizontal lines. The rate of convergence and the selection of the overrelaxation parameters are discussed.

Each implicit step requires the solution of a linear algebraic system with pentadiagonal {authors use "quidiagonal"!} matrix, which the authors solve by elimination. If there are $(M-1)^2$ interior nodes in the grid, each iterative step (double sweep) requires approximately $25(M-1)^2$ multiplication times on a digital computer. When $M=21$, the authors needed 24 double sweeps, which took 54 seconds on a machine multiplying 4000 times per second, to reduce the error to 10^{-6} of its initial value.

The authors claim this method to be decidedly superior to other proposed relaxation methods.

G. E. Forsythe (Stanford, Calif.)

4549:

Birkhoff, Garrett; and Varga, Richard S. Implicit alternating direction methods. Trans. Amer. Math. Soc. **92** (1959), 13–24.

The authors study alternating direction iteration methods of Peaceman, Rachford, and the reviewer for solving elliptic difference equations arising from differential equations of the form $\Sigma u-\nabla\cdot(D\nabla u)=S$, $\Sigma(x)\geq 0$, $D(x)>0$. They show that the method of estimating the convergence rate given by the originators is applicable only if $\Sigma(x)$ and $D(x)$ are constants, the region is a rectangle, and the boundary condition is $u_n+ku=\delta(x, y)$, $k\geq 0$. This result demonstrates one of the reviewer's errors. They then show that, if the parameter ρ of the Peaceman-Rachford method is held fixed and if Σ is a constant, the spectral radius of the method is less than one for any connected region and arbitrary $D(x)>0$. The authors fail to make clear the distinction between a failure of a method of

proof, which is their main point, and the failure of the numerical procedure, which has not happened in the several thousand examples with variable coefficients that the reviewer has observed. Their results point to the need of a penetrating analysis to be applied to alternating direction methods. *J. Douglas, Jr.* (Houston, Tex.)

4550:

Guggenheim, E. A. The accurate numerical solution of the Poisson-Boltzmann equation. Trans. Faraday Soc. **55** (1959), 1714–1724.

"The Poisson-Boltzmann equation formulated by Debye and Hückel [Physik Z. **24** (1923), 185–206] has been accurately solved numerically and the result compared with the solution in series obtained by Gronwall [Proc. Nat. Acad. Sci. U.S.A. **13** (1927), 198–202]."

Author's summary

4551:

Greenspan, D. On 'best' nine-point Laplace difference analogues on rectangular grids. Quart. J. Mech. Appl. Math. **12** (1959), 111–116.

The question is to find the most accurate "9-point" approximation to the Laplace equation $\Delta u \equiv u_{xx}+u_{yy}=0$ in the form $L(u)\equiv\sum_{i=0}^{8}\alpha_i u_i=0$, where $u_0=u(0,0)$, $u_1=u(h,0)$, $u_2=u(0,d)$, $u_3=u(-h,0)$, $u_4=u(0,-d)$, $u_5=u(h,d)$, $u_6=u(-h,d)$, $u_7=u(-h,-d)$, $u_8=u(h,-d)$. Let $p=h/d$. The author finds a solution $\alpha_0=-20$, $\alpha_1=\alpha_3=2(5-p^2)(1+p^2)^{-1}$, $\alpha_2=\alpha_4=2(5p^2-1)(1+p^2)^{-1}$, $\alpha_5=\alpha_6=\alpha_7=\alpha_8=1$. For harmonic functions u the author proves that, when $p\neq 1$, $L(u)=O(h^6)$, as $h\to 0$, and that no higher-order approximation is possible with these 9 points. The usual case $p=1$, when $L(u)=O(h^8)$, had been treated earlier [Greenspan, J. Franklin Inst. **263** (1957), 425–430; MR **19**, 66]. *G. E. Forsythe* (Stanford, Calif.)

4552:

Greenspan, Donald. On the numerical solution of Dirichlet problems. Quart. J. Mech. Appl. Math. **12** (1959), 117–123.

This paper offers a slight extension from a square grid to a rectangular grid of part of the fundamental paper of Gerschgorin [Z. Angew. Math. Mech. **10** (1930), 373–382] bounding the discretization error in solving the Dirichlet problem by finite differences. {The present bound for the discretization error is a trivial special case of the general treatment by Wasow [Z. Angew. Math. Phys. **6** (1955), 81–97; MR **18**, 236], which is not referred to.}

G. E. Forsythe (Stanford, Calif.)

4553:

Greenspan, Donald. Note on difference equation approximations of Laplace's equation. J. Franklin Inst. **268** (1959), 46–52.

There exists a certain linear difference expression $L(u)$, involving with each point only its eight nearest neighbors in a rectangular grid, such that $L(u)=O(h^5)$ for all harmonic functions $u(x, y)$. Here h is the mesh length in the x-direction. The author shows that there is no other difference expression with this property, if the mesh ratio is kept constant as h tends to zero.

W. Wasow (Madison, Wis.)

4554:

Milnes, Harold W.; and Potts, Renfrey B. Boundary contraction solution of Laplace's differential equation. J. Assoc. Comput. Mach. **6** (1959), 226–235.

The authors propose a certain marching process for solving difference equations corresponding to the Dirichlet problem for Laplace's equation in a plane region G interior to a simple closed curve. The aim is to avoid having to store at one time the functional values for the full set of grid points in G. The process is developed for a circular disk G, and there is no discussion of practically extending it to more general regions.

If G is the circle $\{(r, \theta)|r\leq r_0\}$, grid points are chosen on a set of concentric circles with radii $r_{j+1}=Rr_j$ $(0<R<1;$ $j=0, 1, \cdots)$. On the circle $r=r_j$, $2N$ grid points are spaced at equal angular intervals $\Delta\theta=\pi/N$. The authors approximate the Poisson integral by a certain sum, and thus express the vector u_{j+1} of values at the grid points on $r=r_{j+1}$ as Au_j, where A is a circulant matrix independent of j.

The authors consider a "simple"mesh in which grid-points of all circles $r=r_j$ lie on the same $2N$ rays $\theta=k\pi/N$ $(k=0, 1, \cdots, 2N-1)$, and a "staggered" mesh in which every other ring of grid points is rotated by the angle $\Delta\theta/2$. They show that the dominant eigenvalue of A is asymptotically $1-2R^{2N}$ for the staggered mesh, and $1+2R^{2N}$ for the simple mesh, as $R\to 1$. They conclude that the simple mesh is unstable, but that the staggered mesh is stable. Satisfactory experience with the staggered mesh on an IBM704 is cited, with $2N=50$ and $R=0.7$. Some ways of reducing the arithmetic are described.

G. E. Forsythe (Stanford, Calif.)

4555:

Douglas, Jim, Jr. Round-off error in the numerical solution of the heat equation. J. Assoc. Comput. Mach. **6** (1959), 48–58.

A 2-point boundary-value problem for the heat equation with prescribed end-point values is modelled by the implicit difference equations

$$\Delta_x{}^2 w_{i,j+1} = (w_{i,j+1}-w_{ij})(\Delta t)^{-1},$$

$$w_{1,j+1} = d_{1,j+1},\ w_{n,j+1} = d_{n,j+1}, \tag{*}$$

where

$$\Delta_x{}^2 w_{i,j+1} = (w_{i+1,j+1}-2w_{i,j+1}+w_{i-1,i+1})(\Delta x)^{-2}.$$

The author carefully studies the growth of round-off error in solving the system (*) by a form of Gaussian elimination for each value of the time index j. If ε is a bound for the round-off error in a single arithmetic operation, it is proved that the round-off error at a fixed time $T=j\Delta t$ is bounded by $L_5\in(\Delta t)^{-1}$, for all values of Δt, where L_5 depends on $(\Delta x)^2/\Delta t$ and on the initial and boundary conditions. The author claims without much detail that this very stable behavior is attributable to a certain normalization of each equation at each stage of the elimination solution of the tridiagonal implicit system (*). {Since the analysis purports to hold for both fixed- and floating-point operation, the reviewer fails to understand the vital importance of normalization in controlling round-off errors.}

G. E. Forsythe (Stanford, Calif.)

4556:

Sheldon, J. W. Algebraic approximations for Laplace's

equation in the neighborhood of interfaces. Math. Tables Aids Comput. **12** (1958), 174–186.

Let C_1 be an analytic arc separating plane regions G_1 and G_2. The author is concerned with finite-difference methods for solving plane potential problems in $G_1 \cup G_2$—i.e., to find $V^{(i)}$ harmonic in G_i $(i=1, 2)$ with $V^{(1)}=V^{(2)}$ on C_1 and $\xi(\partial V^{(1)}/\partial n) = \partial V^{(2)}/\partial n$ on C_1, where $\xi>0$ is constant and where $\partial/\partial n$ is normal differentiation in the direction from G_1 into G_2. Let a square mesh include a point $a=(0, 0)$ on C_1, and neighbors $b=(h, 0)$, $c=(0, h)$, $d=(-h, 0)$, and $e=(0, -h)$. The main question is how to select a linear algebraic relation $L(V_a, V_b, V_c, V_d, V_e)=0$ to model the above interface conditions.

Suppose C_1 is a straight line segment of inclination α not a multiple of $\pi/4$. The author claims that, for $\xi\neq 1$, no linear combination of $V_a, \cdots, V_e$ will represent the Laplacian ΔV of V on C_1 in such a way that the local truncation error tends to 0, as $h\to 0$. By "local truncation error" the author means the difference on C_1 between $L(V_a, \cdots, V_e)$ and ΔV, for either $V=V^{(1)}$ or $V=V^{(2)}$.

There is discussion of: (1) how to add another neighbor of a to reduce the local truncation error, (2) what to do with mesh points not on C_1, (3) formulas when α is a multiple of $\pi/4$, (4) what to do with curved C_1, and so on. There is some comparison with interface formulas of Douglas, Gardner, and Peaceman [unpublished].

The author remarks that, even though the local truncation error does not vanish, as $h\to 0$, the solution of the difference equations may approach the solution of the potential problem! *G. E. Forsythe* (Stanford, Calif.)

4557:

Zondek, B.; and Sheldon, J. W. On the error propagation in Adams' extrapolation method. Math. Tables Aids Comput. **13** (1959), 52–55.

This paper is a discussion of a trivial case of error-propagation, dressed up in mathematical language which almost entirely obscures its simplicity. The equation considered is reduced by restrictions to a first order linear differential equation with constant coefficients; this offers little difficulty in any case, either for solution or discussion of error propagation. The method of solution considered, the Adams method, is not a good one as it stands; it would not be used without a corrector formula. It is further truncated to a fixed number of terms, thus further restricting its applicability.

J. C. P. Miller (Madison, Wis.)

4558:

★Gouarné, René. Le calcul automatique des déterminants par la méthode des cycles. Actes du colloque de calcul numérique, Périgueux, 1957, pp. 29–38. Publ. Sci. Tech. Ministère de l'Air, Notes Tech. no. 80, Paris, 1959. vii+87 pp. 1800 francs.

The author continues his discussion of the evaluation of determinants by his so-called method of cycles. This method makes it possible to develop a determinant by an ingenious graphical procedure. This procedure lends itself to automatic calculation, and a flow diagram is shown for such a problem.

H. H. Goldstine (Yorktown Heights, N.Y.)

4559:

Sander, Arnold. Rechenschieber für komplexe Zahlen. Math.-Phys. Semesterber. **5** (1956), 128–131 (1 plate).

4560:

Mirsepassi, T. Graphical evaluation of a convolution integral. Math. Tables Aids Comput. **13** (1959), 202–212.

A graphic method based on a finite difference approximation is described for evaluating the convolution integral

$$x(t) = \int_0^t f(t-\tau)g'(\tau)d\tau,$$

given the functions $g(t)$ and $f(t)$. Once a chart has been set up, the value of the integral is obtained by adding a series of readings. The method is particularly convenient if, for a given $g(t)$, the integral is to be evaluated a large number of times.

The same charts may also be used to find $f(t)$ given $x(t)$.

C. Froese (Vancouver, B.C.)

4561:

Chao, F. H. A new method of practical harmonic analysis. Acta Math. Sinica **6** (1956), 433–451. (Chinese. English summary)

A practical method is given for computing approximately the first $2n+1$ Fourier coefficients of a function $f(x)$. The method consists of replacing $f(x)$ under the integral sign by the piecewise linear function connecting the endpoints $(l\pi/n, f(l\pi/n))$, or by the function which connects three consecutive points by parabolic arcs.

Yu Why Chen (New York, N.Y.)

4562:

★Selfridge, R. G.; and Maxfield, J. E. A table of the incomplete elliptic integral of the third kind. Dover Publications, Inc., New York, 1958. xiv+805 pp. $7.50.

In the notation of the author's introductory text, the elliptic integral of the third kind is defined by

$$\Pi(\phi, \alpha^2, k) = \int_0^\phi \frac{d\vartheta}{(1-\alpha^2\sin^2\vartheta)\sqrt{(1-k^2\sin^2\vartheta)}}.$$

For each value of α^2, out of the set

$$\alpha^2 = -1.00(0.05)-0.10(0.02)-0.02,$$
$$0.05(0.05)0.50(0.02)0.80(0.01)0.99,$$

six-decimal values of $\Pi(\phi, \alpha^2, k)$ are given on twelve consecutive pages for full sets of the angular variable $\phi=0.00(0.01)1.57$ and the parameter $k=\sin\theta$, $\theta=0.1(0.1)1.5$. The introduction contains relevant transformation formulae, details of computation, checking, round-off errors, and indications for use of the table.

{Reviewer's remark. The caption to this beautiful table is seriously misleading: On top of each of the 804 pages, α should read α^2, Θ should read ϕ, while the "K values" listed are actually those of k^2. Further, in the footnote reference 3 on page viii the page number should be 223 instead of the apparently wrong 600. Also, the last k on line 5 on page ix should read K, and in the preceding line $1-k^2$ should read $(1-k^2)$.} *C. J. Bouwkamp* (Eindhoven)

4563:

Riley, J. A.; and Billings, C. Gaussian quadrature of some integrals involving Airy functions. Math. Tables Aids Comput. **13** (1959), 97–101.

This paper tabulates the integrals

$$\int_0^\infty x^\lambda \frac{uv}{F^2}\,dx, \quad \int_0^\infty x^\lambda \frac{u'v'}{G^2}\,dx,$$

where $F^2=(\mathrm{Ai}\,x)^2+(\mathrm{Bi}\,x)^2$, $G^2=(\mathrm{Ai}'\,x)^2+(\mathrm{Bi}'\,x)^2$, primes denoting derivatives with respect to x, and where the pairs (u, v) have the particular values $(\mathrm{Ai}\,x, 1)$, $(\mathrm{Ai}\,x, \mathrm{Ai}\,x)$, $(\mathrm{Ai}\,x, \mathrm{Bi}\,x)$, $(\mathrm{Bi}\,x, 1)$. Here $\mathrm{Ai}\,x$, $\mathrm{Bi}\,x$, are the Airy Functions; these are solutions of $y''=xy$. Values are given to 5 figures for $\lambda=0(1)20$ for each of the eight integrals, together with a description of the method of quadrature used. Also given are 9-decimal values of abscissas and weights for the 27-point Gaussian quadrature formulae for the range $(-1, 1)$. *J. C. P. Miller* (Madison, Wis.)

COMPUTING MACHINES

See also 4538, 4543, 4824.

4564:

★Alt, Franz L. Electronic digital computers: their use in science and engineering. Applied Mathematics and Mechanics, Vol. 4. Academic Press Inc., New York-London, 1958. x+336 pp. $10.00.

The first two chapters of this book present in the usual way general facts about computer logic, such as number representation, arithmetic of computers, sequencing and also historical remarks. In chapter 3 the reader gets acquainted with the coding system of two existing machines, namely the 4-address-system of the SEAC and the 1-address-system of the IBM 704. Several examples are given there as well as general advice how to organize computations, though chapter 3 is not intended as an instruction manual. Automatic programming is mentioned only briefly.

The most interesting part is chapter 4 which deals mainly with numerical analysis, though the treatment of the various subjects such as error analysis, evaluation of explicit functions, differentiation and integration, numerical solution of ordinary and partial differential equations, algebraic problems is of course not exhaustive. In this chapter the author does not confine himself to the usual representation but gives many interesting remarks concerning pitfalls and peculiar situations which may arise.

The book closes with chapter 5 which is an account of problems which have been solved by computers. There one can also find a very interesting section (5.3) discussing the organisation of a computing center.

H. Rutishauser (Zürich)

4565:

★Tomovic, Rajko. Direct programming of digital computers. Symposium on modern computers: Proceedings of the Rome Symposium (17–18 October 1956) organized by the Preparatory Committee of the International Computation Center under the auspices of Unesco, pp. 16–22. Libreria Eredi Virgilio Veschi, Rome. vii+94 pp.

The author notes that computer programming would be much simplified if quantities could be referred to by an identifier rather than by the address which stores them. It is suggested that machines be built (a) with distinct memories for instructions and numbers, and (b) with numbers having identifiers attached to them whenever they are sorted.

When a number is wanted it is called by its identifier and the machine searches through the store until it locates the appropriate identifier.

{Reviewer's remark: The common technique of symbolic addressing, with an assembly program for substituting real addresses for symbolic ones in a preliminary pass, achieves the desired result without the extra time or hardware needed for searching.}

C. C. Gotlieb (Toronto, Ont.)

4566:

★Walther, A. Vergleichende Beispiele für den Gebrauch moderner Rechenautomaten. Symposium on modern computers: Proceedings of the Rome Symposium (17–18 October 1956) organized by the Preparatory Committee of the International Computation Center under the auspices of Unesco, pp. 4–15. Libreria Eredi Virgilio Veschi, Rome. vii+94 pp.

4567:

★Ercoli, Paolo; and Vacca, Roberto. Errors due to overflow in arithmetic operations particularly as regards FINAC electronic computer. Symposium on modern computers: Proceedings of the Rome Symposium (17–18 October 1956) organized by the Preparatory Committee of the International Computation Center under the auspices of Unesco, pp. 30–38. Libreria Eredi Virgilio Veschi, Rome. vii+94 pp.

4568:

Ashenhurst, R. L.; and Metropolis, N. Unnormalized floating point arithmetic. J. Assoc. Comput. Mach. **6** (1959), 415–428.

The authors discuss what they call unnormalized floating point arithmetics or significant digit arithmetics. They propose various arithmetic rules and make error analyses for each. These analyses are statistical in character. The paper closes with a comparison of normalized and unnormalized arithmetics.

H. H. Goldstine (Yorktown Heights, N.Y.)

4569:

Blaauw, G. A. Indexing and control-word techniques. IBM J. Res. Develop. **3** (1959), 288–301.

Certain automatic indexing features are being built into the Stretch computer which simplify the addressing and processing of blocks of data. These features include the conventional index register (or B-line) operations of address modification and counting as well as several novel elaborations of these functions.

A "control word" is a word which is used to describe the location and size of a block of data. Three parts of a control word (called fields) when interpreted as numbers represent (1) the address of the first word of the block, (2) the number of words in the block, and (3) a "refill" field which is the address of the next control word to be used. The first two of these fields may be used to provide the initial setting of an index register which is used to facilitate the processing of the block in question. The first field provides the initial setting of the address for elements within the block, while the second provides the setting for the counter which is used to detect the end of the block. Although the refill address is not used during the processing of the block it permits automatic resetting of the index register after the processing has been completed.

Another use for control word reference techniques is

found in data processing. Control words may be arranged in a chain or even a loop by suitably adjusting the refill address of each control word to refer to the succeeding word in the chain. This ordering of control words implies an ordering of the corresponding blocks of data. During data processing this implied order may be altered by simply interchanging control words rather than by the more time consuming operation of transferring the corresponding blocks of data.

Examples are given of such various uses of control words. *D. E. Muller* (Urbana, Ill.)

4570:

Perlis, A. J.; and Samelson, K. Report on the algorithmic language ALGOL. Numer. Math. **1** (1959), 41–60.

The authors present the report of a joint committee of the Association for Computing Machinery and the Gesellschaft für angewandte Mathematik und Mechanik. The report presents the specifications of an algorithmic language within which numerical procedures may be precisely presented in a standard form. Such a standardized language has two important purposes: 1. It permits direct presentation of a numerical procedure to any suitable machine. 2. It permits precise and detailed communication of numerical procedures among numerical analysts.

Ad 1: A procedure properly stated in ALGOL may be used as input to any digital computer for which an automatic translation program has been prepared which transforms ALGOL into that machine's own "command" language (i.e. code). Thus the "suitable machine" mentioned in the first purpose could be the combination of a machine which does the translating and a machine which does the computing. The machines with "loop control" can do both; the time efficiency of the translation, however, is a function of the size of the rapid access storage. ALGOL is therefore a "universal command language of the algebraic type" in the terminology of the reviewer. The feasibility of such "universal machine codes" was shown in an experiment at Aberdeen Proving Ground in 1954 [J. Assoc. Comput. Mach. **4** (1957), 254–273; MR **19**, 1201].

Automatic translation programs from ALGOL to machine code are now in process of construction for many types of general purpose machines in America and Europe.

Ad 2: Each abstract numerical procedure has, ideally, one "infinite precision numerical analysis". There is for each of these, however, a variety of orderings of component operations, each with a different "finite precision numerical analysis". There are, for example, dozens of operationally different methods of effecting the Newton-Raphson square root procedure, each with different ordering of operations, error effects due to finite precision, discrimination methods determining the decision to terminate iteration, etc. The idealized procedure may therefore be embedded in a routine of commands in many ways, each with its own types of efficiency in accuracy, storage requirements, and running time.

Classical numerical analysts like Runge paid attention to this variety of methods of controlling a procedure; their attention to such details as convenient tabular forms is an example. ALGOL includes not only a quasi-algebraic notation for the presentation of the mathematical procedure, but also a command notation for presenting its "flow of control". IBM's FORTRAN and Sperry Rand's MATHMATIC were examples of such command languages for specific machines.

Numerical analysts will henceforth be able to present with precision, not merely mathematical procedures, but their control as well, in a language which is easily scanned and is furthermore both machine independent and independent of their own national languages.

The plans are to produce a series of "Taschenbücher" during the next few years, which books will include a standard library of numerical procedures in ALGOL; presentation of new procedures will be in ALGOL in such publications as Numerische Mathematik, Communications of the Association for Computing Machinery, etc.

The report under review presents a discursive syntactical description of ALGOL in a late stage of its process of standardization. It distinguishes three types of alphabet in which ALGOL procedures can be presented, which alphabets have, in effect, simple character conversion translations between any one of one type and any one of another. One of these, the so-called "reference language", is to have a uniquely standardized representation. The other two are called the "publication" and the "hardware representation" types. For example, the usual exponent typography "x^y" in a publication language might have the unique typography "$x \uparrow y \downarrow$" in the reference language and a string of four groups of six bit symbols (where the six bit group for $\uparrow$ would ordinarily have designated a symbol like "$") in some binary machine hardware representation language.

The bulk of the report is a syntactical description of the reference language for ALGOL. At the character level are the letters of the roman alphabet, the ciphers of the arabic number system, arithmetic operators and relations, boolean operators, substitution symbols, and various control symbols. The control symbols include punctuation marks, bracketing marks, clause markers like "for" and "if", and type markers like "comment", "array", and "procedure", and finally sequencing control markers like "go to", "stop", "return", etc. We note that at the character level of the reference language are included a handful of complete English words; beside the ones just mentioned we might mention "begin" and "end", which are used as bracketing marks at the sentence level.

The word level of the syntax is defined recursively. Strings of characters of recognizable form designate identifiers (whether of variables, expressions, or sentences), integers, numbers, simple variables, subscripted variables, parameters, functions, statement labels, expressions (whether arithmetic, boolean, or designational), and lists of expressions.

The clause and sentence levels are defined recursively between themselves as suitable strings of word level strings of characters. At the clause level are expressions of the conditional type ("if" followed by a boolean expression) and quantifying expressions indicating by lists the range of values through which a program is to be looped or iterated ("for" followed by a list or other range of value designater). At the sentence level are declarations ("comment" followed by any string of characters, "type" identification, "array" declarations, programmed switch declarations, and definitions of functions and procedures) and commands (assignment of expressions to identified storage, "stop", "return", "go to " sequencing commands, commands for alternative procedures under stated

conditions, commands to carry out a named procedure for stated values of parameters, and commands to copy sections of program with variation by substitution).

Programs in the resulting language are also easy for humans to scan. For example, the authors illustrate the language with a Simpson integration procedure in which the mesh size is automatically changed until successive approximations differ by less than a stated tolerance. The ease with which the program may be read allows one to see immediately that it must fail for periodic functions where the interval of integration is an integral multiple of four times the period. The fact that it is so easy to recognize the limitation of the procedure described speaks well for the clarity of ALGOL.

S. Gorn (Philadelphia, Pa.)

4571:

Bottenbruch, Hermann. Erläuterung der algorithmischen Sprache ALGOL an Hand einiger elementarer Programmierbeispiele. Bl. Deutsch. Ges. Versicherungsmath. **4** (1959), 199–208.

The author refers to the algorithmic language ALGOL reported on by Perlis and Samelson [see preceding review]. He illustrates its use by presenting a number of programs from actuarial mathematics including: (a) a commutation and interest computation; (b) a commutation under change of mortality statistics; (c) a premium computation for a mixed insurance, and, finally, a program involving ten different but similar formulae.

The use of "procedure statements" in ALGOL is illustrated by a commutation procedure common to examples a and b, and the simplification effected in these programs by the use of such a "subroutine".

The last example is used to illustrate a complicated double-indexed programmed switch, and the possibilities made available in ALGOL by the use of a statement calling for a copying with substitutions.

S. Gorn (Philadelphia, Pa.)

4572:

Beneš, R. Über die Symbolik der Programmierung programmgesteuerter elektronischer Rechenanlagen. Mitt. Verein. Schweiz. Versich.-Math. **59** (1959), 59–74. (French, Italian and English summaries)

The current notations for schematic flow charts are fairly close to standardization if one ignores input and output calls and formats. The author suggests a notation equally applicable to all strictly sequential inputs and outputs (e.g. punched cards, punched tape, and magnetic tape with which one does not alternately read and write) and illustrates its use in a fairly complicated automatic sequential control of two inputs. It is clear that his proposal generalizes to any number of strictly sequential inputs and outputs, subject to a strictly sequential control.

The need still exists to extend such standardization to processing systems either with non-strictly-sequential components, such as drums, or which possess the ability to have a number of components operating simultaneously. Among the simplest examples would be "process charts" for systems in which there are several input and output buffers and convertors, in which, in addition, rewinding can proceed independently, and in which there are various intermediate storages with independent controls. Thus small drum machines could be intermediate computers in the system, and could operate their own on-line printers each with its own selective-print-out plug board. More complicated systems would include a number of instruction registers operating simultaneously.

S. Gorn (Philadelphia, Pa.)

4573:

Berry, F. J. Intercode, a simplified coding scheme for AMOS. Comput. J. **2** (1959), 55–58.

Intercode is an interpretive routine for the Ferranti Mark I* computer, providing for 11 three-address pseudo instructions and 100 floating point numbers. It is useful for training programmers and for short computing problems.

C. C. Gotlieb (Toronto, Ont.)

4574:

Windley, P. F. The influence of storage access time on merging processes in a computer. Comput. J. **2** (1959), 49–53.

The time T required to internally sort N items by merging is shown to be

$$T = (\alpha(p-1)+\beta)N[\log_p N],$$

where p is the merging base (i.e., the number of items being compared on a single pass), α is the time taken to compare two items, β is twice the average access time to the backing store, and $[\log_p N]$ is the smallest integer greater than or equal to $\log_p N$. From this it is shown that, for sufficiently large N, the optimum base is given by the smallest p satisfying

$$\frac{p+\lambda}{\log p} < \frac{p+1+\lambda}{\log(p+1)},$$

where $\lambda=\beta/\alpha-1$. Tables illustrating the application of this result are given. The effect of changing the base during the sort is considered (it is sometimes best to use bases p and $(p+1)$ in succession) and also the effect of the time required for block transfers (it is usually negligible).

C. C. Gotlieb (Toronto, Ont.)

4575:

Rind, René. Langages pour machines. Présent et avenir de la programmation automatique. Chiffres **1** (1958), 123–132.

The author describes various degrees of automaticity in the communication between a programmer and his general purpose machine. He estimates that symbolic and assembly programs do about 5% of the work of translation from "human" to "machine" language. The next advance he calls that of programming an "exterior logic", whereby, for example, the machine is programmed to simulate one possessing floating point operations or a three address structure (Grace Hopper's old A-2 compiler would be an example). He makes no sharp distinction here between interpretive and compiling mode of operation, stating that such programs carry us about 25% of the way to direct communication. The third advance he calls "abstraction", wherein the machine interprets symbolic addresses; this phase, he says, carries us about 40% of the road to direct communication. Anything beyond this third phase he calls automatic programming, and cites "macro-instruction", compilers, and Fortran. The paper ends with conjectures about programming of the future.

S. Gorn (Philadelphia, Pa.)

4576:
Matyaš, I. The analog computer set up procedure for solving linear differential equations with variable coefficients. Avtomat. i Telemeh. **20** (1959), 839–847. (Russian. English summary)

"Two methods of solving variable coefficients linear differential equations of the type $\sum_{i=0}^{n} a_i u^{(i)} = \sum_{i=0}^{m} b_i v^{(i)}$, $m \leqq n$, on analog computers are described. The methods are illustrated with examples." *Author's summary*

MECHANICS OF PARTICLES AND SYSTEMS

See also 4267, 4280, 4370, 4386, 4594, 4629.

4577:
Chetaev, N. G. [Četaev, N. G.] On the extension of the optical-mechanical analogy. J. Appl. Math. Mech. **22** (1958), 678–681 (487–489 Prikl. Mat. Meh.).

In this article the author establishes the analogy between the mathematical theory of light according to Cauchy and the stable motions of conservative and holonomic dynamical systems. The holonomic constraints and the external forces are assumed to be independent of time, the latter derived from a potential function U. If $V(q_i, \alpha_i)$ + a constant is the complete integral of the Hamilton-Jacobi equation, it is well known that the general solution of the equation of motion is given by

$$\text{(a)} \quad p_i = \frac{\partial V}{\partial q_i}, \quad \beta_i = \frac{\partial V}{\partial \alpha_i} - t\frac{\partial E}{\partial \alpha_i} \quad (i = 1, 2, \cdots, n),$$

where α_i, β_i are the constants which are usually employed in the Hamilton-Jacobi theory of dynamics. The constant of energy E is assumed to be a function of α_i. Assuming those motions of the system which are stable with respect to the variables and allowing small perturbations in the initial values only, the author has derived from the variation equations of Poincaré, in the coordinates and momenta, the following expressions for the coordinate variations ξ_i

$$\text{(b)} \quad \frac{d\xi_i}{dt} = \sum_{j,s} \xi_s \frac{\partial}{\partial q_s}\left(g_{ij}\frac{\partial V}{\partial q_j}\right) \quad (i = 1, 2, \cdots, n),$$

which for a stable unperturbed motion are reducible by a non-singular linear transformation to a linear system with constant coefficients. The characteristic exponents of the system of independent solutions in such a system are known to have zero values. The variations of the coordinates and momenta are defined by the variation of β_i with fixed α_i. They have characteristic exponents equal to zero, provided the unperturbed motion is a stable one. Introducing a normal system of independent solutions ξ_{ik} of (b), the solutions of the reduced system $x_{ik} = \sum_j \gamma_{ij}\xi_{jk}$ are given by the expression

$$\text{(c)} \quad |x_{ik}| = \Gamma C \exp\left(\int \sum \frac{\partial}{\partial q_i}\left(g_{ij}\frac{\partial V}{\partial q_j}\right)\right)dt, \quad |\gamma_{ij}| = \Gamma.$$

Since the characteristic exponents of the stable unperturbed motion vanish, the integral in (c) must vanish. Thus for a stable perturbed motion the author gets the partial differential equation of the elliptic type which the characteristic function V must satisfy, namely

$$\text{(d)} \quad \sum \frac{\partial}{\partial q_i}\left(g_{ij}\frac{\partial V}{\partial q_j}\right) = 0.$$

By taking a twice differentiable function, $\varphi(V - Et)$, of the complete integral of the Hamilton-Jacobi equation, the following equation for φ has been derived

$$\text{(e)} \quad \frac{\partial}{\partial q_i}\left(g_{ij}\frac{\partial \varphi}{\partial q_j}\right) = \frac{2(E-U)}{E^2}\frac{\partial^2 \varphi}{\partial t^2}$$

for the case of a stable unperturbed motion of the system. Putting the coefficient of the right hand side of (e) equal to the reciprocal of the square of the velocity of propagation, it becomes identical to the wave equation of optics, thus establishing the analogy between the mathematical theory of light of Cauchy and the stable motions of holonomic conservative dynamical system.

On the other hand, if $\varphi(x, t)$ is represented in the form, $\varphi = \psi(x)\exp(-2\pi i E t/h)$, (e) reduces to

$$\text{(f)} \quad \nabla^2\psi + \frac{8^2 m}{h^2}(E-U)\psi = 0,$$

which is the Schrödinger equation for a particle. It is interesting to note that the condition for a stable perturbed motion (d) on the solution of a stable unperturbed system having zero characteristic exponents results in a partial differential equation for a function dependent of the complete integral of the Hamiltonian-Jacobi equation of the system, which can be associated with wave motion similar to that of light or the Schrödinger equation of wave mechanics. *N. Chako* (Flushing, N.Y.)

4578:
Sispánov, Sergio. Motion of a spherical top on a horizontal plane. Acta Cuyana Ingen. **1**, no. 9, 1–16 (1957). (Spanish)

This concerns an inhomogeneous sphere, having two equal principal moments of inertia, which rolls and pivots without slipping on a horizontal plane. A system of differential equations of motion is derived, and it is shown that this system can be solved by quadratures. Some general properties of the motion are discussed briefly.
L. A. MacColl (New York, N.Y.)

4579:
Pignedoli, Antonio. Sul moto centrale di un punto-massa veloce. I. Boll. Un. Mat. Ital. (3) **13** (1958), 341–350. (English summary)

L'A. fa uno studio qualitativo dei moti centrali nell'ambito della teoria della relatività.—Siano m_0 ed M_0 le masse di quiete rispettivamente di un pianeta P (puntiforme) e del sole 0 (puntiforme) e sia r la loro distanza all'istante t. Assumendo come forza interagente fra i due corpi quella newtoniana, se f è la costante di gravitazione (di Cavendish) si ha l'equazione

$$\frac{d}{dt}\left[m\frac{d(P-0)}{dt}\right] = -fm_0\frac{m_0+M_0}{r^3}(P-0),$$

dove $m = m_0(1 - v^2c^{-2})^{-1/2}$. Posto $\mu_0 = fm_0(m_0 + M_0)$, $U = \mu_0/r$, l'equazione differenziale della traiettoria diviene

$$\left(\frac{dr^{-1}}{d\theta}\right)^2 + \frac{1}{r^2} = \frac{1}{A^2c^2}\left[\left(\frac{\mu_0}{r} - K\right)^2 - m_0^2c^4\right]$$

da cui si ricava $r^{-1}=p^{-1}[1+e\cos\lambda(\theta-\theta_0)]$ essendo

$$p=\frac{\mu_0^2-A^2c^2}{K\mu_0},\quad \lambda=\left(1-\frac{\mu_0^2}{A^2c^2}\right)^{1/2},$$

$$e^2=\frac{A^2c^2}{\mu_0^2}\left(1-\frac{m_0^2c^4\lambda^2}{K^2}\right).$$

Affinchè e sia reale, dev'essere $\lambda^2\leqq K^2/m_0^2c^4$. Risulta poi $e\gtreqless 1$ secondo che si ha $m_0^2c^4/K^2\lesseqgtr 1$. Nel caso di $e<1$, la traiettoria è compresa tra le due circonferenze di raggi $r_1=p/(1+e)$, $r_2=p/(1-e)$ ed è una tipica curva a rosetta.

Lo spostamento postivo del perielio vale in radianti

$$\sigma=2\pi\left(\frac{1}{\lambda}-1\right)\sim\frac{\pi\mu_0}{A^2c^2}.$$

L'A. discute anche il moto centrale relativistico in un campo di forza elastica (moto dell'elettrone).

C. Lampariello (Rome)

4580:

Bressan, Aldo. Sulle sollecitazioni Lagrangiane, dipendenti anche dall'atto di moto, e derivanti da un potenziale generalizzato. Rend. Sem. Mat. Univ. Padova **28** (1958), 263–279.

The author considers a dynamical system with n coordinates q_i, the generalized forces Q_i being functions of q, $\dot q$ and t and gives necessary and sufficient conditions for Q_i being derivable from a generalized potential function $V(q,\dot q,t)$ (that means $Q_i=(d/dt)\partial V/\partial\dot q_i-\partial V/\partial q_i$), showing that Q_i must be then a linear function of $\dot q$ of a certain type. A further condition is introduced by supposing that no work is done by Q_i; as a consequence Q_i is a homogeneous linear function of $\dot q$. The question is answered whether the system admits m integrals of momentum. Some systems considered by Grioli [Rend. Sem. Mat. Univ. Padova **27** (1957), 90–102; MR **19**, 899] are shown to be special cases of the author's. An example is added concerning the motion of a rigid body, the force being of the Coriolis type.

O. Bottema (Delft)

4581a:

Yarov-Yarovoĭ, M. S. Expansion of the force function of the Newtonian attraction of two bodies. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **1958**, no. 5, 55–61. (Russian)

4581b:

Yarov-Yarovoĭ, M. S. Convergence of an expansion of the force function of the attraction of two bodies. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **1958**, no. 5, 63–65. (Russian)

To evaluate the gravitational potential energy U of two disjoint solids in space, the distance r between points P_1, P_2 of the two solids is expressed in terms of coordinates with origin at the center of mass and axes the principal axes of inertia of each body. Expansion of r^{-1} in a series in terms of these 6 coordinates and integration yields the desired series for U; the series is shown to converge if $r_0>R_1+R_2$, where r_0 is the distance between the centers of mass and the R_k are upper bounds for the distance of a point of the kth body from its center of mass.

W. Kaplan (Ann Arbor, Mich.)

4582:

Bellman, Richard. Functional equations and maximum range. Quart. Appl. Math. **17** (1959), 316–318.

The author indicates how maximum altitude and range for vertical motions and motions in a plane may be determined by use of functional equations.

H. A. Antosiewicz (Los Angeles, Calif.)

4583:

Zeuli, Tino. Moto di rotolamento di una sfera non omogenea pesante su un piano orizzontale (problema del tippe-top). Univ. e Politec. Torino. Rend. Sem. Mat. **17** (1957/58), 143–159.

L'A. discute in questo lavoro il problema del tippe-top. Il sistema dinamico è costituito da una sfera, di sostanza plastica, non omogenea, leggermente smussata e munita, in corrispondenza alla smussatura, di un piccolo stelo radiale.— Se si imprime al sistema una rapida rotazione poggiandolo su di un piano orizzontale e con lo stelo in alto, esso finisce col capovolgersi e rotare con la parte dello stelo poggiata sul piano orizzontale.

G. Lampariello (Rome)

4584:

Manarini, Mario. Sul centro d'inerzia e sul cono d'inerzia relativi ad un sistema rigido e loro applicazioni alle rotazioni permanenti di moti per inerzia e dei giroscopi. Giorn. Mat. Battaglini (5) **6** (**86**) (1958), 66–110.

In questa Memoria l'A., rifacendosi ad altri lavori suoi intorno ai moti giroscopici, approfondisce in maniera interessante i classici risultati che la dinamica dei corpi rigidi deve ad O. Stande ed a W. van der Woude [cfr. T. Levi-Civita e U. Amaldi, *Lezioni di meccanica razionale*, vol. II, parte 2, nuova ed., Zanichelli, Bologna, 1952; MR **14**, 419]. La terminologia introdotta dall'A. quale quella di centro d'inerzia e di cono d'inerzia (con di Stande) e i metodi vettoriali-omografici usati nella Memoria contribuiscono a rendere perspicua l'intera trattazione della teoria dinamica che ha interesse non solo dal punto di vista teorico, ma anche, non meno, dal punto di vista tecnico.

G. Lampariello (Rome)

4585:

Bressan, Aldo. Sulle precessioni d'un corpo rigido costituenti moti di Hess. Rend. Sem. Mat. Univ. Padova **27** (1957), 276–283.

Certain precessional motions in the case of asymmetric rigid bodies under the influence of gravity were discovered by Grioli [Ann. Univ. Ferrara. Sez. VII. (N.S.) **3** (1954), 31–43; MR **17**, 204]. These precessional motions are special cases of the so-called motions of Hess, which satisfy a certain invariant relationship in the partially integrable case discovered by Hess. The author asks if the motions discovered by Grioli are the only precessional motions of Hess. He gives an affirmative answer in the cases when the axis of spin is assumed to be the line joining the fixed point with the center of gravity or when the angular momentum vector is horizontal.

D. C. Lewis, Jr. (Baltimore, Md.)

4586:

Magnus, K. Zur Entwicklung des Stabilitätsbegriffes in der Mechanik. Naturwissenschaften **46** (1959), 590–595.

STATISTICAL THERMODYNAMICS AND MECHANICS

See also 4264, 4699, 4719, 4720, 4730, 4800, 4803, 4804.

4587:

Morita, Tohru. Expansion formulae of the free energy and distribution functions in powers of the one particle distribution function. Progr. Theoret. Phys. **21** (1959), 501–510.

The equilibrium free energy of a multicomponent nonuniform lattice system is expressed as a virial expansion in powers of $\rho_r(\mathbf{R})$, the number of molecules at $\mathbf{R}$. To obtain this result the author regards identical molecules situated on different lattice sites as belonging to different species, and then introduces a super-lattice, each site of which corresponds to a replica of the original lattice; two molecules interact with each other only if they occupy the same site in the super-lattice. This device permits the application of existing virial expansions for uniform systems [K. Fuchs, Proc. Roy. Soc. London Ser. A **179** (1942), 408–432; MR **4**, 30]. Virial expansions are also given for the potentials of average force as well as for the n-particle distribution function; the transition to continuous systems is readily made by letting the volume of each cell go to zero. *S. Prager* (Minneapolis, Minn.)

4588:

Ikenberry, E. The evaluation of collision integrals, using Grad's representation of the distribution function. Arch. Rational Mech. Anal. **3** (1959), 123–132.

This paper is devoted to exact calculation of collision integrals in the kinetic theory of gases. The author gives first a simplified development of the properties of the n-dimensional Hermite polynomials of Grad [Comm. Pure Appl. Math. **2** (1949), 325–330; MR **11**, 514], in terms of which he uses a series expansion of the distribution function slightly more general than Grad's [ibid., 331–407; MR **11**, 473]. The essential step is the simple proof in Sect. V that explicit calculation of all collision integrals is equivalent to finding the power series expansion of a single definite integral. An eight-fold quadrature is to be evaluated. Six of these quadratures the author is able to evaluate explicitly for any spherically symmetric molecular model. The result, expressed by Eq. (5.32), gives the collision integral of an arbitrary Hermite polynomial as a finite linear combination of certain double integrals.

It has been known since Maxwell's day that all integrations can be performed explicitly for an inverse fifth power law; the author shows that such is the case also for rigid spheres and exhibits the result in full generality. For an intermolecular force varying as the inverse nth power, the author is able to carry out one of the two remaining quadratures explicitly.

The explicit formulae given in this paper provide important weapons for possible future attack on the exact theory of the Boltzmann equation.

C. Truesdell (Bloomington, Ind.)

4589:

Cole, G. H. A. Structure of nonuniform liquids. Amer. J. Phys. **27** (1959), 545–554.

In an earlier paper [same J. **27** (1959), 136–146] the modern arguments which lead to an understanding of a simple classical liquid under uniform conditions were considered. The more general case of nonuniform liquids is treated in the present paper and the methods currently used to determine the physical structure under these conditions are explained. *Author's summary*

4590:

Morita, Tohru. Theory of classical fluids: hyper-netted chain approximation. II. Formulation for multi-component systems. Progr. Theoret. Phys. **21** (1959), 361–382.

In a previous paper [same Progr. **20** (1958), 920–938; MR **21** #1722] the author introduced the hyper-netted chain approximation in the cluster expansions for free energy and radial distribution function of a gas. This approximation is here extended to multi-component systems and is applied to ionic solutions. A comparison is given with other theories of these systems.

L. Van Hove (Utrecht)

4591:

Orlov, A. N.; and Men', A. N. The statistical theory of bond energy in spinel-type crystals. Soviet Physics. Solid State **1** (1) (1959), 174–181 (195–202 Fiz. Tverd. Tela).

The statistical (Thomas-Fermi) theory of electron distributions in crystal lattices is used for a study of the binding energies of spinel-type crystals. In these substances the relatively large oxygen ions are interspersed with various metallic atoms of smaller size. The characteristic properties of the crystals depend largely on the interactions of the metallic atoms, while the oxygen ions play a relatively inert role. Starting from this physical conception the authors replace the oxygen ions by an effective potential function, and then study the electron density distribution around the metallic atoms by use of the Thomas-Fermi equation. Two different models of the effective potential function are discussed. The authors are able to give a general account of the dependence of the binding energy and the lattice constant on the composition of the crystal. *E. L. Hill* (Minneapolis, Minn.)

4592:

Mazur, P.; and Postma, B. J. On the molecular theory of the Kerr effect. Physica **25** (1959), 251–267.

The author presents a statistical mechanics for a dense fluid of anisotropic, non-polar molecules subjected to external static and oscillatory electric fields. The polarizability of the molecules is assumed to be independent of molecular interactions and the static field. The Kerr effect, the electrical birefringence induced by an external field, is computed as a function of the polarizability tensor and the density of the medium.

G. Newell (Providence, R.I.)

4593:

Mogyoródi, József. Probabilistic treatment of the motion of neutrons in nuclear reactors. Magyar Tud. Akad. Mat. Kutató Int. Közl. **3** (1958), 237–250. (Hungarian. Russian and English summaries)

A neutron with initial energy E_0 is slowing down in an infinite medium containing atomic nuclei of many different types. Colliding with the nuclei, the neutron loses its energy by scattering or by being absorbed. It is supposed that the free path of the neutron has an exponential distribution with a parameter depending on the energy of

the neutron and, further, that the scattering of the neutron is isotropic. After scattering, the energy of a neutron, with initial energy E, will have a uniform distribution in the interval $[E((A-1)/(A+1))^2, E]$ where A is the mass number of the nucleus involved. The author determines the joint probability that the neutron reaches the energy level E_0e^{-x} $(0<x<\infty)$ without absorption and that the total path-length of the neutron until its energy decreases to E_0e^{-x} is $\leq z$. *L. Takács* (New York, N.Y.)

4594:

Toda, Morikazu. On the Brownian motion of a classical oscillator. J. Phys. Soc. Japan **14** (1959), 722–728.

The Brownian motion of a classical harmonic oscillator is treated from a mechanical standpoint, the oscillator being coupled to a large number of other harmonic oscillators which form the thermostat. By means of a suitable transformation, which eliminates the coupling to first order, the second order quantities, such as the frictional force, are obtained. A discussion is given of the correlation of the random force. *S. Simons* (London)

ELASTICITY, PLASTICITY

See also 4833.

4595:

★Godfrey, D. E. R. Theoretical elasticity and plasticity for engineers. Thames and Hudson, London, 1959. x+311 pp. (2 plates) 42s.

The main virtue of this book is that it is easily read. It might be described as a collection of simple solutions of the simplest equations of elasticity and plasticity. Chapter titles provide a brief description of the contents. These are: Analysis of stress, Analysis of strain, Pure bending of a beam, Torsion of a beam, Bending of a beam with shear, Two dimensional elasticity, Problems in curvilinear coordinates, Concentrated and discontinuous loads, Theory of thin flat plates, Theory of plasticity, Plastic bending and torsion of a beam, Plane problems of plasticity. There are two appendices covering pertinent mathematical topics.

Discussions of the foundations of elasticity and plasticity theories are held to a very bare minimum. General theorems and results are omitted. In short, the beginner seriously interested in solid mechanics would be well advised to spend his time studying a less superficial work. This book may be useful as a reference work since it is easy to read one part without frequent reference to what has preceded it. *J. L. Ericksen* (Baltimore, Md.)

4596:

Noll, Walter. A mathematical theory of the mechanical behavior of continuous media. Arch. Rational Mech. Anal. **2** (1958), 198–226.

This paper presents a logical derivation from explicitly stated assumptions of the constitutive equation governing the mechanical behaviour of deformable materials. The stress at a particle of the material is assumed to be determined by the past history of the deformation of an arbitrarily small neighbourhood of the particle. Using the principle that the stress at the particle is unaltered by a uniform translation of the stress and deformation along the spatial and time axes, it is shown that the stress must be a hereditary functional of the deformation gradients at all times up to and including the instant at which it is measured. The restrictions which may be imposed on the form of this functional, by the consideration that the constitutive equation is unchanged by a rigid rotation of the physical system considered, are obtained. The constitutive equation so derived is that previously obtained by Green and Rivlin [same Arch. **1** (1957), 1–21; MR **20** #2130], although the method of derivation is different. Various special forms of the constitutive equation, some novel, are given. *R. S. Rivlin* (Providence, R.I.)

4597:

Coleman, Bernard D.; and Noll, Walter. On certain steady flows of general fluids. Arch. Rational Mech. Anal. **3**, 289–303 (1959).

The authors continue earlier work [Noll, #4596 above] on a general class of fluids exhibiting hereditary effects in which the stress is assumed to be a functional of a given tensor-valued function which specifies the history of the deformation. Attention is confined to incompressible fluids, and for these it is shown that certain problems involving steady flow can be solved in complete generality without restricting the form of the constitutive equations. The motions considered are: rectilinear flow, including simple shearing flow and flow through a channel (parallel infinite plates at rest), Poiseuille flow and Couette flow. In each case solutions are given for the stresses and these solutions are obtained in terms of three unspecified real functions of a single real variable.

J. E. Adkins (Nottingham)

4598:

Colonnetti, Gustavo. Nuovi punti di vista sul problema dell'equilibrio elasto-plastico nel tempo. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **25** (1958), 140–145.

This note contains some qualitative remarks on one-dimensional theories of Volterra's type, in which "any feature of the deformation" is a functional of the deformation. [Rigorous and invariant theories of this type have been proposed and explored by Green and Rivlin, Arch. Rational Mech. Anal. **1** (1957), 1–21; MR **20** #2130; Noll, #4596 above; and Green, Rivlin and Spencer, ibid. **3** (1959), 82–90; MR **21** #2418.]

C. Truesdell (Bloomington, Ind.)

4599:

Kulikovskiĭ, A. G. On media in which unidimensional motions with homogeneous deformations are possible. Dokl. Akad. Nauk SSSR **120** (1958), 485–486. (Russian)

The medium, subject only to internal pressures, is to move according to $r=\mu(t)r_0$, where r is the distance of a particle from a plane, an axis, or a point, and r_0 is the initial value of r, under either the adiabatic condition $\partial S/\partial t=0$ or the uniform temperature condition $\partial T/\partial r=0$. The author considers equations of state under which this is possible. *F. V. Atkinson* (Canberra)

4600:
de Géry, Jérôme Chastenet. Une représentation intrinsèque simple du tenseur d'énergie de déformation (cas anisotrope) par des opérateurs linéaires de l'espace à trois dimensions. C. R. Acad. Sci. Paris **248** (1959), 1765–1768.

Using the language of affinors, the author obtains an invariant form for the strain energy in linear elasticity and discusses its relation to material symmetries.

C. Truesdell (Bloomington, Ind.)

4601:
Caricato, Gaetano. Sulle trasformazioni di un sistema elastico atte a conservare l'isotropia. Rend. Mat. e Appl. (5) **17** (1958), 313–318.

Consider an elastic material which is isotropic with respect to two different reference configurations but not with respect to all. The author shows that the one reference configuration is related to the other by a conformal transformation. *C. Truesdell* (Bloomington, Ind.)

4602:
Sparacio, Renato. Il cerchio di Mohr del tensore simmetrico del 2° ordine. Rend. Accad. Sci. Fis. Mat. Napoli (4) **25** (1958), 31–45.

4603:
Baldacci, Riccardo F. Sopra un principio variazionale nella teoria degli spostamenti elastici non infinitesimi. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **23** (1957), 250–254.

Viene considerato il problema principale della elastostatica nella ipotesi che non vengano trascurati i termini quadratici nelle espressioni delle componenti di deformazione, restando però queste legate alle componenti dello sforzo dalla classica legge di Hooke. Si costruiscono due funzionali J e K tali che la condizione di stazionarietà per J nella classe delle configurazioni congruenti equivalga, sotto opportune ipotesi di derivabilità, alla condizione di equilibrio e, reciprocamente, la condizione di stazionarietà per K si riduca, sotto analoghe ipotesi di derivabilità, alle equazioni di congruenza.

G. Fichera (Madison, Wis.)

4604:
Urbanowski, Wojciech. Small deformations superposed on finite deformation of a curvilinearly orthotropic body. Arch. Mech. Stos. **11** (1959), 223–241. (Polish and Russian summaries)

Equations are developed using a general form of strain energy for a curvilinear orthotropic body. The general equations are used to study two special cases (i) small deformations superposed on uniform finite extensions of a rectilinearly orthotropic body and (ii) a small twist superposed on a curvilinearly orthotropic tube subjected to a finite inflation and extension.

A. E. Green (Newcastle-upon-Tyne)

4605:
Engl, Walter. Der Spannungszustand einer Reihe von Kräften, die in der unendlichen Halbebene angreifen. Z. Angew. Math. Mech. **39** (1959), 192–198. (English, French and Russian summaries)

The author considers the problem of determining the stress in a semi-infinite plate which is acted on by an infinite series of concentrated forces, all uniformly spaced and orthogonal to the plate boundary. The solution is assumed to be expressible in terms of a function of a hyper-complex number. The boundary conditions are shown to be satisfied by use of residue theory (for both the stress components and the displacements). Various special cases of the theory are discussed. Finally, a particular numerical example is analyzed.

N. Coburn (Ann Arbor, Mich.)

4606:
Paria, Gunadhar. The method of Wiener-Hopf in elastic problems. Bull. Calcutta Math. Soc. **49** (1957), 37–42.

This paper shows how the Wiener-Hopf technique can be applied to the solution of a problem in two-dimensional elasticity concerning an infinite strip of elastic material $0 \leq y \leq 1$, $-\infty < x < \infty$. The edge $y=1$ is loaded by distributed normal pressure only, while the edge $y=0$ is free from traction in the range $x<0$ but is held against a rigid wall (i.e., has zero normal displacement) in the range $x>0$. It is assumed that the loading is such that equilibrium is maintained. The solution of the equations of elastic equilibrium under these conditions is reduced by means of a Fourier transform to that of a Wiener-Hopf problem of standard type. The solution of the problem is given in detail but not in a form suitable for computation.

I. N. Sneddon (Glasgow)

4607:
Bufler, Hans. Einige strenge Lösungen für den Spannungszustand in ebenen Verbundkörpern. Z. Angew. Math. Mech. **39** (1959), 218–236. (English, French and Russian summaries)

The state of stress in a two-dimensional semi-infinite elastic solid in contact with a finite rigid solid, or another semi-infinite elastic solid, along a finite length of its boundary is investigated. Two coupled singular integral equations are formulated which involve the tractions in the normal and tangential directions at the contact length. Exact solutions are found for these tractions as well as the stress along the boundary; they are then applied to five special cases: normal load, shearing load with twisting, shearing load without twisting, twisting moment, and thermal and initial stresses.

Yi-Yuan Yu (Brooklyn, N.Y.)

4608:
Teodorescu, P. P. Quelques considérations concernant le problème plan de la théorie de l'élasticité. Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) **1 (49)** (1957), 369–376.

Equazioni della elasticità piana nelle componenti degli sforzi considerate nella ipotesi che queste siano legate da particolari relazioni lineari a coefficienti costanti.

G. Fichera (Madison, Wis.)

4609:
Golecki, Józef. On the assumption of incompressibility in plane problems of the theory of elasticity. Arch. Mech. Stos. **11** (1959), 297–301. (Polish and Russian summaries)

4610:

Mossakovskii, V. I. Pressure of a circular die [punch] on an elastic half-space, whose modulus of elasticity is an exponential function of depth. J. Appl. Math. Mech. **22** (1958), 168–171 (123–125 Prikl. Mat. Meh.).

A solution, which is better suited for practical computations, is given of the problem of Korenev [Dokl. Akad. Nauk SSSR **112** (1957), 823–826] which concerns the punching by a rigid die of a semi-infinite solid whose modulus of elasticity varies with depth according to an exponential law. Certain inaccuracies in the Korenev paper are also corrected.

The basic pair of dual of integral equations is solved by use of Mellin transforms. As an example, the pressure of a plane circular die on an elastic half-space of this type is considered. *I. N. Sneddon* (Glasgow)

4611:

Fredricks, R. W. Solution of a pair of integral equations from elastostatics. Proc. Nat. Acad. Sci. U.S.A. **44** (1958), 309–312.

The dual integral equations under consideration are

$$\int_0^\infty \xi^{-1}P(\xi)K(\xi x)d\xi = g(x), \quad 0 \leq x \leq a,$$

$$\int_0^\infty P(\xi)K(\xi x)d\xi = 0, \quad x > a,$$

where $K(\xi x)$ is either $\cos(\xi x)$ or $\sin(\xi x)$. In the sine case it is shown that the solution is

$$P(\xi) = 2\sum_{n=0}^{\infty}\sum_{q=1}^{\infty}(2q+1)D_n J_{q+1}(n\pi)J_{q+1}(a\xi),$$

where the constants D_n are the Fourier sine coefficients of $g(x)$ in the interval $0 \leq x \leq a$. A similar expression is derived in the cosine case. The convergence of the series involved is investigated. This analysis covers an important practical case in which the conditions for the validity of the normal Titchmarsh-Busbridge solution are violated. *I. N. Sneddon* (Glasgow)

4612:

An, Bunzai. The elasticity problem of a rigid sphere pressed on the plane surface of a semi-infinite elastic solid. (The effect of friction on the contact surface.) Bull. JSME **2** (1959), 244–251.

Displacements and stresses are expressed by one biharmonic function. An application of the Hankel transform leads to dual integral equations, the solution of which is known. An attempt is made to simplify the final results. Friction is accounted for by applying a procedure equivalent to the multiplication of the horizontal displacement on the contact surface in the non-friction case by a constant coefficient.

The paper is obscured by many typographical errors. *H. Zorski* (Warsaw)

4613:

Dutt, S. B. Torsion of a circular cylinder of varying rigidity with isotropic core. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **93** (1958/59), 117–123.

In this paper expressions are determined for the deformations and stresses at any point in a circular cylinder with rigidity varying with the radius surrounding an isotropic core, under the action of tractions distributed over the boundary. Special cases mentioned include (i) constant traction over a part of the outer surface, (ii) hollow cylinder, (iii) rigid inclusion. *R. M. Morris* (Cardiff)

4614:

Chakravorti, A. Flexure of a composite beam whose section is bounded by intersecting arcs of non-concentric circles. J. Tech. Bengal Engrg. Coll. **3** (1958), 99–108.

Under a load symmetrically applied at one end, the title problem is solved by using bipolar coordinates and Fourier integrals. Different Young's moduli but same Poisson's ratio are taken for the two materials of the beam. The stresses are obtained in infinite integral forms. *S. C. Das* (Madras)

4615:

Solomon, Liviu. Observations sur la flexion d'une console sous l'action d'une sollicitation excentrique. Com. Acad. R. P. Romîne **8** (1958), 1235–1242. (Romanian. Russian and French summaries)

Consider a prismatic beam rigidly supported in the $\xi\eta$-plane, the other end at $z=l$ being free. Let the ξ and η axes be parallel to the principal axes of inertia of cross-sections $z=$ const., and let the centroids of these sections be at ξ_0, η_0, z, where $\xi_0 \neq 0$ in general. Suppose a force R_1 parallel to the ξ-axis is applied at 0, 0, l. The standard treatment for force applied at the centroid of the free end is adapted to the present case of eccentric loading. In particular, in the stress component $\sigma_{33} = k(l-z)(\xi-\xi_0)$ the constant $k = -R_1/J_2$ is independent of ξ_0, where J_2 is the principal moment of inertia of a section about $\xi=\xi_0$. On the other hand, as one would expect, the torsional twist per unit z depends on ξ_0. *J. H. Giese* (Aberdeen, Md.)

4616:

Solomon, Liviu. De la flexion sans torsion d'une console symétrique. Com. Acad. R. P. Romîne **8** (1958), 1011–1019. (Romanian. Russian and French summaries)

Consider a prismatic beam rigidly supported in the xy-plane, with the other end free, and whose cross-sections parallel to $z=0$ have axes of symmetry parallel to the x-axis. Suppose a force parallel to the x-axis is applied at the centroid of the free end. Then by use of general properties of plane harmonic functions the author proves that the beam bends without torsion. In particular the twist at the free end is proportional to

$$A = \iint (x\chi_y - y\chi_x + x\phi_2 - y\psi_1)dxdy$$

extended over the cross section $\mathcal{D}$ of the beam, where the harmonic function determines the bending stresses, and ψ_1 and ψ_2 are known functions such that $x\psi_2 - y\psi_1$ is an odd function of y. By Gauss's theorem $A = -\oint \chi(\partial\phi/\partial n)$ is taken around the boundary C of $\mathcal{D}$, where the harmonic function ϕ determines the torsional stresses. Since $\partial\phi/\partial n$ is a known odd function of y on C, then to prove $A=0$ it suffices to show $\chi(x,-y) = \chi(x,y)$. The author does this by showing that the Green's function of $\mathcal{D}$ satisfies $G(x,-y;p,-q) = G(x,y;p,q)$, and that by virtue of the boundary conditions for χ the conjugate harmonic function of χ is an odd function of y. *J. H. Giese* (Aberdeen, Md.)

4617:
Bryukker, L. È.; and Kuršin, L. M. Derivation by statical methods of the bending equations of three-layered plates with rigid additions. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk. Meh. Mašinostr. **1959**, no. 3, 167–168. (Russian)

The equilibrium equations for the stress resultants of a triple-layer sandwich plate are derived from simple statics. The core material is assumed to have no rigidity against deformations in the plane of the plate but to be capable of carrying transverse shears, bending, and twisting moments. It is claimed that the pairwise equality of transverse shear stresses of the core cannot be enforced if the equilibrium equations for the stress resultants of the triple-layer plate are to have the same form as obtained by variational methods. *F. T. Geyling* (Summit, N.J.)

4618:
Balmer, Hans A. Stress functions for plates of convex polygonal shape. J. Aero/Space Sci. **26** (1959), 675–676.

4619:
Jung, Hans. Über die Bestimmungen von Eigenspannungen in Scheiben. Wiss. Z. Hochsch. Schwermaschinenbau Magdeburg **2** (1958), 45–50.

4620:
Černina, V. S. Stressed state of a torus-shaped shell of intermediate thickness. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk. Meh. Mašinostr. **1959**, no. 3, 96–104. (Russian)

The author addresses himself to the stress analysis of shells with intermediate thickness such as found in turbine housings. He uses the theories of Reissner and Naghdi where the solution is separable into a first approximation and corrections whose magnitudes are expressible in powers of h/a where h = thickness of shell and a = some typical radius of curvature. Reissner's theory is briefly reconstructed and carried on to the two so-called "Meissner equations" in terms of two specially defined dependent variables. For the case of toroidal shells this system is reduced to one differential equation in terms of a complex solution and a numerical example is worked out for explicit comparison between the theory of thin shells and the present improved analysis.
F. T. Geyling (Summit, N.J.)

4621:
Reissner, E. On torsion of thin cylindrical shells. J. Mech. Phys. Solids **7** (1959), 157–162.

"In the following we consider an application of the theory of elastic, thin, cylindrical shells to the problem of St. Venant torsion. We obtain in this way simple explicit general results for non-homogeneous shells, which include as special cases Bredt's formulas for closed tubes and known results for tubes slit along a generator. We also find that the problem of St. Venant torsion requires use of a formulation of shell theory which is more accurate than Love's first approximation theory." (Author's summary) *J. W. Craggs* (Newcastle-upon-Tyne)

4622:
Teodorescu, Petre P. Sur le problème plan de l'élasticité de certains corps anisotropes. VII. Les corps actionnés par des forces massiques quelconques. Com. Acad. R. P. Romîne **8** (1958), 887–892. (Romanian. Russian and French summaries)

4623:
Teodorescu, Petre P. Sur le problème plan de l'élasticité de certains corps anisotropes. VIII. L'influence des variations de la température. Com. Acad. R. P. Romîne **8** (1958), 1119–1126. (Romanian. Russian and French summaries)

4624:
Teodorescu, Petre P. Sur le problème plan de l'élasticité de certains corps anisotropes. IX. Le cas des petits mouvements élastiques. Com. Acad. R. P. Romîne **8** (1958), 1243–1250. (Romanian. Russian and French summaries)

4625:
Chakravorty, Saktikanta. Disturbances due to localized periodic surface traction in a semi infinite elastic medium with transverse isotropy. Riv. Mat. Univ. Parma **8** (1957), 375–379.

The transverse isotropy being about a line (z-axis) perpendicular to the free boundary (xy-plane), solutions corresponding to Lamb's problem [Philos. Trans. Roy. Soc. London Ser. A **203** (1903), 1–42] have been obtained in integral forms, under an isolated periodic force at the origin. *S. C. Das* (Madras)

4626:
Choudhury, P. Stresses due to a certain type of nucleus of strain in an infinite slab of transversely isotropic material. Riv. Mat. Univ. Parma **8** (1957), 337–343.

Nucleus of strain being in the form of a centre of rotation situated inside the material and along the axis of elastic symmetry (taken as z-axis), one plane face is taken as rigidly fixed, while the other, the xy-plane, is free. The problem is reduced to finding a single stress function and solutions have been obtained by superimposing a stress system to satisfy the boundary conditions. Hankel inversion theorem is used and some numerical results have been given. *S. C. Das* (Madras)

4627:
Choudhury, Pritindu. Stresses in an elastic layer with varying modulus of elasticity. Bull. Calcutta Math. Soc. **49** (1957), 99–104.

The author considers the distribution of stress in a two-dimensional layer $0 \leq y \leq b$, $-\infty < x < \infty$, when the normal component of stress on $y=0$ is defined to be $-P$ when $|x| < a$ and zero otherwise, and the shearing stress on $y=0$ is zero. On $y=b$ it is assumed that both components of the displacement vector vanish. It is assumed that the Poisson ratio is constant throughout the layer but that the Young's modulus varies as $\exp(ky)$. The basic equations are solved by means of a Fourier transform in

x, the expressions for the components of stress being given in the form of trigonometrical integrals. The variation of the edge stress on the line $y=0$ is calculated numerically.

I. N. Sneddon (Glasgow)

4628:

Davin, M. État de contrainte et de déformation d'un certain milieu hétérogène indéfini à deux dimensions. Arch. Mech. Stos. **11** (1959), 129–156. (Polish and Russian summaries)

This paper gives an iterative process for finding the Airy Stress function in a plane heterogeneous medium consisting of two different isotropic media which is made up in the following way. The one medium occupies the interior of all circles having the same radius unity and whose centres are situated at the nodal points of a square mesh of given dimensions, whilst the second material occupies the space outside these circles. The infinite plane is taken to be under a given tension at infinity and successive biharmonic terms in plane polar coordinates are used in an iterative way to satisfy the required conditions at each boundary circle between the two media.

R. M. Morris (Cardiff)

4629:

Neuber, H. Über allgemeine Eigenschaften der Schwingungszahlen linear-elastischer Systeme. Ing.-Arch. **28** (1959), 229–241.

The paper considers the general properties of the natural frequencies of complex, linear, elastic systems, consisting of springs and masses. General expressions for

$$\sum_{r=1}^{m} \omega_r^2, \quad \sum_{r=1}^{m} 1/\omega_r^2, \quad \sum_{r=1}^{m-1} \sum_{s=r+1}^{m} \omega_r^2 \omega_s^2, \text{ etc.},$$

where ω_r is a natural frequency of the system, are given in terms of the natural frequencies of simple hypothetical sub-systems. The analysis is extended to continuous systems by considering the vibrations of bars.

G. B. Warburton (Edinburgh)

4630:

Di Taranto, R. A. Natural frequencies of nonuniform beams on multiple elastic supports. J. Appl. Mech. **25** (1958), 57–63.

The method for finding the natural frequencies of a non-uniform beam is to approximate the given beam by means of a fictitious beam in which the mass is concentrated at n points. The weightless portion between two consecutive points of concentrated mass is analyzed in such a manner as to yield recurrence relations in passing from one point to the next. The final equation for the determination of the frequencies is obtained in the usual manner by equating to zero the determinant of a system of homogeneous linear equations.

D. C. Lewis, Jr. (Baltimore, Md.)

4631:

Teodorescu, Petre P. Über die Berechnung des periodisch belasteten elastischen Halbraumes. Rev. Méc. Appl. **4** (1959), no. 1, 141–148. (1 insert)

4632:

★Handelman, George; Boyce, William; and Cohen, Hirsh. Vibrations of a uniform, rotating beam with tip mass. Proceedings of the Third U.S. National Congress of Applied Mechanics, Brown University, Providence, R.I., June 11–14, 1958, pp. 175–180. American Society of Mechanical Engineers, New York, 1958. xxvii+864 pp. $20.00.

A uniform rod is considered rotating with constant angular velocity about an axis through the clamped end and perpendicular to the rod. Small lateral vibrations of the rod in a plane through the rod and containing the axis of rotation, that is normal to the plane of rotation of the rod, are studied. A tip mass of negligible moment of inertia is attached to the free end. The differential equation, and boundary conditions are developed, and the variational principles governing the natural frequencies and modes of vibration are established. Bounds on the natural frequencies are obtained and presented for various values of the parameters prescribing the problem. Interesting relationships are obtained in this way for special circumstances.

The centrifugal tension in the rod influences the free end boundary condition, and the reviewer feels that such an effect should also be included in the differential equation.

E. H. Lee (Providence, R.I)

4633:

Wegner, Udo. Ein Beitrag zu den Stabilitätskriterien der Elastizitätstheorie. Ing.-Arch. **28** (1959), 357–359.

The paper describes a formal analogy between stability criterion in the theory of elasticity and wave front theory. The author intends to give more complete development in a future paper.

W. T. Koiter (Delft)

4634:

Mansfield, E. H. On the buckling of certain optimum plate structures with linearly varying thickness. Aero. Quart. **10** (1959), 145–148.

"This paper is concerned with the buckling under uniform longitudinal compression of a variety of structures composed of plates whose thickness tapers linearly to zero across the section. For given cross-sectional area and overall dimensions the sections with linearly varying thickness achieve a greater buckling load (assuming that local buckling, rather than Eulerian buckling, is the criterion) than sections with any other smooth variation of thickness. These particular sections are therefore optimum sections and, even if they may not be used in practice, provide a convenient yardstick for purposes of comparison. The buckling loads are considerably greater than those for the corresponding 'constant thickness' sections." (From the author's summary)

W. T. Koiter (Delft)

4635:

Capriz, Gianfranco. A proposito di una classificazione dei problemi di stabilità elastica. Riv. Mat. Univ. Parma **8** (1957), 371–374. (English summary)

4636:

Romashov, A. N.; Rodionov, V. N.; and Sukhotin, A. P. On an explosion in an unbounded compacting medium. Soviet Physics. Dokl. **123** (3) (1958), 1283–1286 (627–630 Dokl. Akad. Nauk SSSR).

"The unusual propagation of an explosion in soil is connected with the large irreversible deformations of the

medium. The present paper gives basic experimental and theoretical results of the investigation of an explosion in an inelastic compacting medium." *Authors' summary*

4637:

Fleishman, B. A. Progressing waves in an infinite nonlinear string. Proc. Amer. Math. Soc. **10** (1959), 329–334.

The non-linear one-dimensional wave equation $u_{tt}-u_{xx}=-\alpha u-\beta u^3$ admits bounded and continuous solutions of the form $u=\psi(x-ct)$ provided that $|c|\neq 1$. They can be expressed in terms of Jacobian elliptic functions. For $\alpha\beta>0$, all such solutions are periodic, with $|c|>1$ if $\alpha>0$, $\beta>0$, and $|c|<1$ if $\alpha<0$, $\beta<0$. For $\alpha\beta<0$, both periodic and non-periodic solutions exist, with both $|c|>1$ and $|c|<1$. In this case there are also discontinuous waves that travel at unit speed.

F. G. Friedlander (Cambridge, England)

4638:

Samuels, J. Clifton. Reflection and refraction of elastic waves at the interface of two moving semi-infinite plane media. J. Acoust. Soc. Amer. **31** (1959), 1076–1079.

4639:

Flügge, W.; and Zajac, E. E. Bending impact waves in beams. Ing.-Arch. **28** (1959), 59–70.

The authors consider the Timoshenko beam equation and give four methods of obtaining approximate solutions using either the Laplace transform or the Fourier sine transform. The solutions are valid in different regions of the space-time plane, and together cover the whole plane. The methods are applied to a special problem concerning wave propagation in a semi-infinite beam, and some numerical results are given.

G. Eason (Newcastle-upon-Tyne)

4640:

Babuška, Ivo. Lineare Theorie der inneren Reibung. Apl. Mat. **4** (1959), 177–202. (Russian. Czech and German summaries)

The linear theory internal friction in metals leads the author to study an equation of the form $\sigma=E[(\mathbf{D}\varepsilon)^{\cdot}+\varphi(\varepsilon)]$; σ and ε denote real-valued functions of a real variable, $\mathbf{D}$ is a linear operator on the space C_l of all continuous periodic functions of period l, φ is a linear functional on this same space, and E is a constant (Young's modulus). This equation is a linear relation between a periodic deformation $\varepsilon(t)$ and the corresponding strain $\sigma(t)$ and is sufficiently general to include the various hypotheses that have been introduced in the linear theory. The derivative $(\mathbf{D}\varepsilon)^{\cdot}$ in the equation may not exist. This difficulty is overcome by introducing a generalized notion of solution. The author investigates existence and uniqueness and some properties of these solutions. In particular, he considers the forced oscillations of a system with one degreem of freedom. This appears to be the first of a series of papers. *J. P. LaSalle* (Baltimore, Md.)

4641:

Segawa, Wataru. Rheological equation of Voigtian material. J. Phys. Soc. Japan **14** (1959), 1102–1106.

The author develops a theory of deformation of isotropic (Voigt) materials which exhibit elastic and viscous effects. He assumes the total stress to be the sum of an elastic and a viscous component. The former is derived from the Helmholtz free energy which is assumed to be a function of the strain components alone; the latter component is derived from a dissipation function which depends only upon the velocity gradients. The resulting equations may be regarded as a special case of those considered by Rivlin and Ericksen [J. Rational Mech. Anal. **4** (1955), 323–425; MR **16**, 881] and other workers. The theory is applied to the problems of simple elongation and simple shear. *J. E. Adkins* (Nottingham)

4642:

Šemyakin, E. I. On a method of integration of nonstationary linear boundary value problems on the propagation of disturbances in non-ideally elastic media. Prikl. Mat. Meh. **22** (1958), 289–300. (Russian. English translation in J. Appl. Math. Mech. **22** (1958), 401–416).

A discussion of waves generated by transient surface stress distributions in isotropic solids with dissipation effects. The latter are represented by an operator $[\lambda' K_t, \mu' K_t]$ added to the usual elastic constants of Hooke's law. Some general results are proved, e.g., the solution for time dependent stresses and for a quiescent solid at $t\leq 0$ is given in terms of the corresponding solution for a non-dissipative solid and of the solution $R(t,\tau)$ of an auxiliary differential equation

$$(1+\omega K_t)\frac{\partial^2 R}{\partial \tau^2}=\frac{\partial^2 R}{\partial t^2}$$

These general results are applied to the viscoelastic half space ($K_t=\partial/\partial t$) and to the case with elastic afterworking (K_t an integral operator).

In an introductory section the general equations of motion for dissipative models are discussed. It is unfortunate that the recent and important work of Biot is not mentioned. *I. Tolstoy* (Dobbs Ferry, N.Y.)

4643:

Yamamoto, Misazo. Phenomenological theory of visco-elasticity of three dimensional bodies. J. Phys. Soc. Japan **14** (1959), 313–330.

While the author claims to replace existing visco-elastic theories by a better one, the reviewer is not sure that any definite proposal is made. Equation (37) seems to be only an alternative form of the classical finite elastic stress-strain relations. In the special flows studied, the stresses seem to be determined by special additional hypotheses. While some of the ideas and results are interesting, the reviewer is unable to follow the line of thought.

C. Truesdell (Bloomington, Ind.)

4644:

★Prager, William. An introduction to plasticity. Addison-Wesley Publishing Co., Inc., Reading, Mass.-London, 1959. viii+148 pp. $9.50.

The author's book covers the two disciplines of mathematical and engineering theory of plasticity. He develops, in the first two chapters, methods which can be applied to examples drawn from mechanical and from civil engineering.

The use of the concept of plastic potential makes the book a delight to read. The theory is developed at the end of the first chapter, and is led up to by a series of elegant models of plastic materials. The second chapter discusses the mechanical behaviour of plastic structures, and this behaviour is represented pictorially in what strictly may be called vector diagrams. But they are even more than vector diagrams; looking at one of the author's illustrations of stress space, the reader can almost immediately visualize the collapse mechanism of a given structure, or can convince himself that shakedown will occur under a given range of loading.

The third, and longest, chapter of the book, on Limit analysis and design, deals with applications of plastic theory to a whole range of structures, including portal frames, arches, flat plates, cylindrical shells, and even voussoir arches. Again, in each example discussed, the pictorial representation of the behaviour is an immediate aid both to understanding the problem and to grasping the solution. The chapter ends with some examples of minimum weight design. The fourth and last chapter deals with problems of finite plastic deformation, and, after principles are established, examples are given of sheet extrusion, wedge indentation, tube drawing and so on.

A valuable part of the book is the set of examples given for the reader to work, in which important applications are apparent. The book itself is based on a series of lectures given in Zurich; these were published in Switzerland in 1955 [W. Prager, *Probleme der Plastizitätstheorie*, Birkhäuser Verlag, Basel und Stuttgart, 1955; MR **17**, 917]. It is evident that considerable rewriting has been done since then; later references are not simply added as footnotes, but are integrated into the text. There are no spare lines and there is no padding; the author has considered every word, and thought deeply on every aspect of plastic theory.

If a criticism of the presentation can be made, it is that, paradoxically, the material is too easy for the reader. Occasionally there are certain difficulties that the author does not mention, perhaps because the mathematical treatment of a problem does not fit his adopted simple pattern based on the plastic potential. In some places it is obvious that he has had to use advanced mathematical techniques to arrive at a certain result; the result is presented without the reasoning, and is streamlined into the form chosen for the book. Small digressions might have been welcome; they would, if nothing else, have warned the reader against being over-confident.

But these are essentially trivial criticisms; the author is completely master of this, his main field of study, and he communicates this sense of mastery to the reader.

J. Heyman (Cambridge, England)

4645:

Clavuot, Christian; und Ziegler, Hans. Über einige Verfestigungsregeln. Ing.-Arch. **28** (1959), 13–26.

The mechanical behavior of a rigid, work-hardening solid is specified by (1) the virgin yield condition, (2) the flow rule, and (3) the hardening rule. (1) indicates the states of stress at which plastic flow sets in under loading from the virgin state; (2) establishes a relation between the increments of stress and plastic strain; and (3) specifies the effect of plastic flow on the yield locus. The present paper is concerned with hardening rules proposed by Prager (1949) and Ziegler (1959). For the purpose of this review, these rules will be labelled P and Z, respectively. In a nine-dimensional stress space, in which the rectangular Cartesian stress components are used as the rectangular Cartesian coordinates of the stress point, P states that an infinitesimal increment of plastic strain produces a translation of the yield locus in the direction of its exterior normal at the stress point, whereas according to Z the translation occurs along the radius vector from the center of symmetry of the yield locus to the stress point. When by the nature of a problem certain stress components vanish identically, the hardening process can be discussed in the appropriate linear subspace of stress space. It is shown that Z remains valid in this subspace whereas, in general, P does not. Assuming the virgin yield locus to be given by Mises' yield condition, the authors investigate conditions for the equivalence of P and Z. For plane strain and for plane stress, each of the hardening rules is combined with the yield conditions of Mises (M) and Tresca (T), and certain difficulties arising for the combination PT are pointed out. {The hardening rule P was suggested by the remark that under it the solution of the typical boundary value problem for elastic, work-hardening solids can be shown to be unique; it does not seem that a similar general uniqueness theorem can be established under the hardening rule Z. Since, however, the two rules are equivalent in certain circumstances, adoption of Z cannot generally preclude uniqueness. A characterization of problem types for which Z furnishes unique solutions would be of interest.}

W. Prager (Providence, R.I.)

4646:

Eason, G. The load carrying capacities of cylindrical shells subjected to a ring of force. J. Mech. Phys. Solids **7** (1959), 169–181.

An unsupported circular cylindrical shell of finite or infinite length is subjected to a ring of force. The problem has been treated previously by Drucker [Proc. 1st Midwest. Conf. Solid Mech., 1953, pp. 158–163, Engineering Experiment Station, Univ. of Illinois, Urbana, Ill., 1954] and Eason and Shield [J. Mech. Phys. Solids **4** (1955), 17–27; MR **17**, 805], using an approximate yield condition. Here the exact maximum shearing stress (Tresca) condition is used.

Depending upon whether the load is very close, moderately close, or sufficiently far from either end independently, the solution takes on five different forms. Complete formulas for determining the stress fields are given for each case; velocity formulas are given for two cases and indicated for the other three. Typical results for each case are shown graphically.

P. G. Hodge, Jr. (Chicago, Ill.)

4647:

Grigor'ev, A. S. Large deflections of a rectangular membrane. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk. Meh. Meh. Mašinostr. **1959**, no. 3, 105–113. (Russian)

Plastic deformation of a membrane under uniform pressure is treated by finite difference methods. The Hencky-von Mises yield condition is assumed.

R. C. T. Smith (Armidale)

4648:

Lenskiĭ, V. S.; and Fomina, L. N. Propagation of one-dimensional waves in materials with delayed yield. Izv.

Akad. Nauk SSSR. Otd. Tehn. Nauk. Meh. Mašinostr. **1959**, no. 3, 133–136. (Russian)

One end of a long straight uniform bar is suddenly subjected to an axial stress, non-decreasing in time. Analysis based upon characteristic methods is given of this problem when the material is elastic-plastic and exhibits delayed yield of an assumed simple type. Discussion of results obtained is given in relation to other published work on dynamic loading phenomena in metals.

H. G. Hopkins (Sevenoaks)

4649:

Bland, D. R. Vector fields associated with plane plasticity. Quart. Appl. Math. **14** (1956), 321–323.

4650:

Thomas, T. Y. Fracture of circular pipes by internal pressure. J. Math. Mech. **8** (1959), 623–641.

First part of the paper recapitulates in tensor form the known elementary solution with axial symmetry for a fully plastic open-ended tube under internal pressure. The solid is non-hardening rigid/plastic with Tresca yield function and Mises potential. Second part considers possibility of another solution which is not axially symmetric: on one side of an axial plane the tube remains rigid, on the other side it deforms. The proposed solution involves a tangential discontinuity in velocity across the interface; author interprets this as indicating that the pipe would fracture there. Reviewer notes that the interface is not a surface of maximum shear stress, a fact which in standard plastic theory would be regarded as inconsistent with the proposed tangential velocity discontinuity.

R. Hill (Nottingham)

4651:

Reckling, K.-A. Zur Theorie der Plattenbeulung im plastischen Materialbereich. Ing.-Arch. **28** (1959), 263–276.

In the first half of the paper, the author discusses total and incremental stress-strain laws for elastic, work-hardening materials and surveys the present stand of the theory of plastic plate buckling stressing the fact that, under the assumption of initial isotropy, the physically unsound total laws furnish buckling loads in closer agreement with the experimental evidence than those obtained from the incremental laws. An incremental law for an orthotropic material is proposed, and the associated buckling theory is developed. No detailed comparison between the new theory and experiments is offered beyond the statement that a plausible degree of anisotropy may reduce the buckling load by as much as 20%.

W. Prager (Providence, R.I.)

4652:

Eršov, L. V. Approximate solution of axi-symmetric elasto-plastic problems. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk. Meh. Mašinostr. **1959**, no. 3, 139–142. (Russian)

Discussion is given of an approximate method of solution of certain problems of elastic-plastic deformation in non-hardening ductile metals under quasi-static conditions of axial symmetry. It is assumed that the so-called full plasticity condition of Haar and von Kármán is satisfied in plastic regions. Essentially, the method is based upon the representation of the components of stress and strain as infinite power series in a small parameter, suitably defined by the particular problem being investigated.

Comparatively little progress has been made in applications of theories of axially-symmetric elastic-plastic deformation. The present type of approach appears to have intrinsic limitations. In this reviewer's opinion, substantial progress is only likely to be achieved through developments allied to the work by R. T. Shield [Proc. Roy. Soc. London Ser. A **233** (1955), 267–287; MR **17**, 687] which is based upon explicit recognition of the mathematical type of the governing equations.

H. G. Hopkins (Sevenoaks)

4653:

Librescu, Liviu. Sur le problème thermo-élastique des coques minces, de forme arbitraire. Com. Acad. R. P. Romîne **8** (1958), 809–815. (Romanian. Russian and French summaries)

The author shows that when thermal stresses are considered the partial differential equations for equilibrium of thin shells are unchanged except for the following formal modification. To the terms that involve the external forces one must add known appropriate thermal stress terms which depend on the temperature distribution and the geometry of the shell.

J. H. Giese (Aberdeen, Md.)

4654:

Sternberg, Eli; and Muki, Rokurō. Note on the expansion in powers of Poisson's ratio of solutions in elastostatics. Arch. Rational Mech. Anal. **3**, 229–234 (1959).

Method of expansion is used to study the solution of the three-dimensional classical elastostatics problem for the case of zero body forces and prescribed surface tractions. The individual terms of the expansion admit a physical interpretation by means of steady-state thermoelasticity theory and permit certain conclusions concerning the dependence upon the elastic constants of solutions to space problems. For a body of arbitrary connectivity all stresses are independent of the elastic constants if and only if the dilatation (and the mean stress) is a linear function of the cartesian coordinates. The expansion method does not appear to have any value in the practical solution of problems.

A. E. Green (Newcastle-upon-Tyne)

4655:

Knops, R. J. A method for solving linear thermoelastic problems. J. Mech. Phys. Solids **7** (1959), 182–192.

Consider two solutions S and S' of the isothermal field equations of classical elasticity theory, corresponding to a region D, the same boundary conditions and body forces, the same shear modulus μ, but to two distinct values ν and ν' of Poisson's ratio. The author observes that the difference between two such solutions characterizes the thermal stresses and deformations arising in a medium occupying D which has elastic constants μ and ν is subjected to the appropriate homogeneous boundary conditions, and is under the influence of a temperature field that is proportional to the first stress-invariant of S'. [This observation is closely related to a result by Sternberg and Muki which appeared concurrently; see preceding review.]

As intimated by the author, the chief shortcoming of the foregoing scheme for deducing new solutions to (uncoupled) thermoelastic problems from known solutions to isothermal boundary-value problems in elasticity theory lies in the fact that the temperature fields thus accidentally encountered are, in general, not likely to be of physical interest. In the present note the scheme described is used to re-establish certain known steady-state thermoelastic solutions for the half-space and for a body bounded by two parallel planes, which had previously been established in a constructive manner. The author asserts that his derivation of these results is both shorter and more direct. The second claim is less convincing than the first. *E. Sternberg* (Providence, R.I.)

4656:

Ignaczak, Józef. Thermal displacement in a non-homogeneous elastic semi-infinite space caused by sudden heating of the boundary. Arch. Mech. Stos. **10** (1958), 147–154. (Polish and Russian summaries)

The problem considered is the calculation of the components of the displacement vector in an elastic semi-space composed of two layers $0 \leq x \leq R$, $R \leq x$ with different mechanical and thermal properties. The deformation is assumed to be due to the semi-space being heated suddenly over the whole boundary $x=0$, which is rigidly fixed (i.e., $u=\partial u/\partial t=0$ for all t on $x=0$). It is further assumed that the displacement, stresses, temperature and heat flux are continuous on the boundary $x=R$ separating the two media.

Because of the nature of the boundary conditions only one space-variable x is necessary to specify the problem. The equations of motion and the heat conduction equation are solved by introducing a Laplace transform with respect to the time variable b. The solutions are obtained in closed form but no numerical results are exhibited.

I. N. Sneddon (Glasgow)

4657:

★Flügge, W.; and Conrad, D. A. Thermal singularities for cylindrical shells. Proceedings of the Third U.S. National Congress of Applied Mechanics, Brown University, Providence, R.I., June 11–14, 1958, pp. 321–328. American Society of Mechanical Engineers, New York, 1958. xxvii+864 pp. $20.00.

With the familiar assumptions of shallow shell theory and the additional one that in thermal problems the temperature is known on both surfaces of the shell and varies linearly along a normal through the thickness, the equations of the cylindrical shell are derived. Attention is then given to singular solutions of these equations. A basic set of singular solutions involving Kelvin's ber-, bei-, ker- and kei-functions is derived by a process of separation of variables in the four second-order partial differential equations into which the eighth order equation for the normal component of displacement can be decomposed. The physical interpretation of these singular solutions is discussed in the paper and in two appendices. In Appendix I it is shown that the vertical resultant of the stresses, derived from any one of the set of basic singular solutions, acting on an infinite strip of arbitrary width is zero—implying that no linear combination of the stated solutions can possibly represent a concentrated load applied at the origin. In Appendix II it is verified that the postulated solutions represent "hot spots" (extremely localized heated areas). In addition to being useful in the discussion of the effects of hot-spots on cylindrical shells, these solutions constitute the singular part of the Green's function for a problem of this kind, from which solutions may be developed for arbitrary temperature distributions on the shell surfaces.

I. N. Sneddon (Glasgow)

4658:

Lessen, M. Thermoelastic waves and thermal shock. J. Mech. Phys. Solids **7** (1959), 77–84.

The field equations of linear thermoelasticity are derived and the properties of thermoelastic waves discussed. The outlines of the solution of the problem of determining the stress distribution in a semi-infinite solid when its plane boundary is subjected to a thermal "shock" are then given. No details are given in the present paper though we are informed that "a few examples of interest are now being investigated and calculated in detail".

On the basis of these calculations the author asserts that in the case of "strong" thermal shocks at a free surface, where the mechanical properties of the medium begin to degenerate at the excited surface, the fact that the mechanical and thermal waves are so phased as to produce zero surface stresses may cause failure to occur in the body of the solid where the properties of the medium are intact but the phasing of the two wave trains is such as to produce high stresses. It should, however, be kept in mind that the equations upon which the calculations are based are derived on the assumption that the temperature fluctuations are sufficiently small to enable us to neglect the temperature variation of the physical characteristics of the body. It is difficult to see therefore what is meant by a "strong" thermal shock in this context.

I. N. Sneddon (Glasgow)

4659:

Teodorescu, P. P. On the plane problem of thermoelasticity. Rev. Méc. Appl. **3** (1958), no. 3, 333–340. (Russian)

Author considers the form of the classical equations of statical thermoelasticity when the temperature field is uniquely defined by the function $T(x, y)$, which is independent of the third cartesian coordinate z. It is shown the principal stresses σ_x, σ_y satisfy the equation

$$\Delta(\sigma_x+\sigma_y+kET) = 0,$$

where Δ denotes the two-dimensional Laplacian operator and k and E are constants which, in the case of plane stress, are the coefficient of linear expansion of the solid and its Young's modulus respectively and which, in the case of plane strain, are simple multiples of these quantities.

It is then shown that the Airy stress-function of such problems can be written as $F+F_1$ where $\Delta^2 F=0$ and $\Delta F_1=-kET$.

Similar results are derived in the "dynamical" case, although in the author's equations the operator $\partial/\partial t$ occurs only in the equation governing the conduction of heat in the solid. This equation is the "classical" one and with it are used not the equations of motion of the solid but the equations of statical equilibrium. The results may be taken therefore to refer only to certain types of quasi-static problems.

The paper ends with a discussion of the statical case when there are sources of heat in the solid so that T is governed by the equation $\Delta T + M = 0$, where $M(x, y)$ is a source function. It is pointed out that, in this case, F_1 is a solution of the fourth order equation $\Delta^2 F_1 = EkM$.

The paper consists entirely in showing that the solution of the equations of plane thermoelastic problems can be reduced to the solution of certain partial differential equations (of the type cited above for F and F_1). It does not attempt the solution of any particular boundary value problems nor does it attempt to show the relation between the method presented and the complex-variable approach to such problems. *I. N. Sneddon* (Glasgow)

4660:

Chowdhury, P. Thermal stresses due to uniform temperature distributed over a band of the cylindrical hole in an infinite body. Appl. Sci. Res. A **8** (1959), 474–478.

4661:

Piechocki, Władysław. The stresses in an infinite wedge due to a nucleus of thermoelastic strain. Arch. Mech. Stos. **11** (1959), 211–221. (Polish and Russian summaries)

4662:

Piechocki, Władysław. The state of stress in a circular disc due to the action of a nucleus of thermoelastic strain. Arch. Mech. Stos. **11** (1959), 287–295. (Polish and Russian summaries)

STRUCTURE OF MATTER

See also 4800, 4803, 4804.

4663:

Ginzburg, V. L.; and Pitaevskii, L. P. On the theory of superfluidity. Soviet Physics. JETP **34** (7) (1958), 858–861 (1240–1245 Ž. Eksper. Teoret. Fiz.).

A theory is given for the properties of liquid helium in the neighborhood of the transition point to the superfluid state (λ-point). By analogy with quantum mechanical theory the hydrodynamic and thermodynamic properties of the superfluid component are described in terms of an "effective wave function". Only the case in which the normal component of the fluid is at rest is considered. It is assumed as a boundary condition that the density of the superfluid component vanishes on the boundaries of the liquid. It is shown that the theory can give an account of the lowering of the transition point in liquid films and of vortex lines. *E. L. Hill* (Minneapolis, Minn.)

4664:

Pitaevskii, L. P. Phenomenological theory of superfluidity near the λ point. Soviet Physics. JETP **35** (8) (1959), 282–287 (408–415 Ž. Eksper. Teoret. Fiz.).

Equations given earlier by V. L. Ginzburg and L. P. Pitaevskii [see preceding review] for He II (liq.) in the immediate neighborhood of the λ-point are extended to include the possibility of variation of the density of the superfluid component with time. As in the earlier work, the superfluid component is described by means of a complex-valued function, which is of the nature of a quantum-mechanical density amplitude function. This function is supposed to obey an equation of the form, $\hat{L}\psi = i\hbar\partial\psi/\partial t$, where $\hat{L}$ is a non-Hermitian operator. The hydrodynamic equations are determined by considerations based on energy conservation, momentum conservation, and entropy flux. *E. L. Hill* (Minneapolis, Minn.)

FLUID MECHANICS, ACOUSTICS

See also 4596, 4597, 4598, 4641, 4743, 4751.

4665:

Jaiswal, N. K. Application of finite Hankel transform to a problem in perfect fluid flow. Bull. Calcutta Math. Soc. **50** (1958), 84–86.

A new finite Hankel transform is introduced with the properties that it can be used to remove the terms $\partial^2 v/\partial r^2 + r^{-1}\partial v/\partial r - n^2 v/r^2$ from a partial differential equation and is appropriate to boundary value problems in which the normal derivative $\partial v/\partial r$ vanishes over the surfaces of the cylinders $r = a$, $r = b$.

The method is applied formally to the following boundary value problem: Between two infinite concentric cylinders of radii a and b ($> a$) there is a perfect liquid containing a ring source of radius c which is emitting a volume Q of the liquid per unit time. The velocity potential is left in the form of a Fourier-Bessel series. No attempt is made to obtain a closed form for the series or to calculate it (or the corresponding expressions for the components of velocity) in any particular case. *I. N. Sneddon* (Glasgow)

4666:

Todor, Liviu I. Recherches sur le problème de Poincaré-Stekloff. Acad. R. P. Romîne. Stud. Cerc. Mat. **9** (1958), 503–545. (Romanian. Russian and French summaries)

L'article contient deux chapitres dans lesquels l'A. aborde de deux différentes manières le problème de Poincaré-Stekloff, notamment de déterminer un champ vectoriel dans un domaine tridimensionnel D, lorsqu'on connaît son rotationnel, sa divergence et sa composante normale sur Fr D. Ce problème a été résolu [voir L. Lichtenstein, *Grundlagen der Hydromechanik*, Springer, Berlin, 1929; ou V. Crudeli, Rend. Circ. Mat. Palermo **58** (1934), 166–174] par la resolution d'un nombre élevé des problèmes de Neumann. Dans le 1er chapitre l'A. montre qu'on peut résoudre ce problème par des quadratures et par la résolution d'un seul problème de Neumann. Dans le 2ème chapitre on étudie le problème de la résolubilité de l'équation intégrale de H. Villat [C. R. Acad. Sci. Paris **188** (1929), 837–839] sans supposer que cette équation provient effectivement d'un problème de Poincaré-Stekloff. On en obtient des conditions nécessaires et suffisantes. *C. Foiaş* (Bucharest)

4667:

Dragoș, Lazăr. Sur un mouvement fluide barotrope. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **24** (1958), 142–148.

4668:

Borg, S. F.; and Russomano, R. Stagnation point pressures on symmetrical blunt-nosed bodies entering an infinite incompressible ocean. Z. Angew. Math. Mech. **39** (1959), 246–247.

4669:

★Handbook of supersonic aerodynamics. Vol. 3. Section 7. Three-dimensional airfoils. Navord Report 1488. U.S. Government Printing Office, Washington, D.C., 1957. iv+83 pp. (not consecutively paged). $1.50.

A summary, mainly in graphical form, of the results of linearized supersonic flow theory for the lift, pitching moment, and drag of uncambered wings having double-wedge or biconvex profiles and a variety of polygonal planforms. *M. D. Van Dyke* (Stanford, Calif.)

4670:

Stark, Valter J. E. A method for solving the subsonic problem of the oscillating finite wing with the aid of high-speed digital computers. Svenska Aeroplan A. B. Tech. Note **TN 41**, 36 pp. (1958).

The author formulates an integral equation for the streamwise integral of the acceleration potential for an oscillating wing in subsonic, compressible flow. He then expands the solution in a set of suitable functions and sets up a scheme for determining the expansion coefficients on a digital computer. Results are given for a rectangular wing of aspect ratio 3 in incompressible flow.

J. W. Miles (Los Angeles, Calif.)

4671:

Ionescu, M.; et Darie, Gh. Les caractéristiques aérodynamiques d'une certaine classe d'ailes minces en écoulement homogène d'ordre supérieur. Acad. R. P. Romîne. Stud. Cerc. Mec. Apl. **10** (1959), 367–377. (Romanian. Russian and French summaries)

4672:

Patraulea, N. N. Une méthode simplifiée pour le calcul de l'aile à volet fluide. Acad. R. P. Romîne. Stud. Cerc. Mec. Apl. **10** (1959), 351–365. (Romanian. Russian and French summaries)

4673:

Carafoli, Elie; et Săndulescu, Şerban. Caractéristiques aérodynamiques des ailerons ayant un mouvement oscillatoire harmonique en régime supersonique. Acad. R. P. Romîne. Stud. Cerc. Mec. Apl. **10** (1959), 13–40. (Romanian. Russian and French summaries)

The problem of a quadrilateral aileron oscillating about a hingeline is reduced, by superposition, to the treatment of triangular ailerons, for linearized supersonic flow. Using the quasi-steady approximation appropriate for low reduced frequencies [Miles, J. Aero. Sci. **16** (1949), 378–379; MR **10**, 755] analytic expressions for the aerodynamic coefficients are given. Results are presented for subsonic and supersonic edges. The general formula covers practically all aileron shapes.

N. Rott (Ithaca, N.Y.)

4674:

Tick, Leo J. A non-linear random model of gravity waves. I. J. Math. Mech. **8** (1959), 643–651.

Probabilistic procedures are combined with the perturbation techniques of classical hydrodynamics to obtain the spectrum, cross spectra, and covariance function for a train of long crested progressive gravity waves in deep water. This wave motion satisfies the appropriate non-linear hydrodynamic equations to second order. A perfectly general random representation that satisfies the linearized equations is first obtained. This representation is then specialized so that the underlying linear system becomes a stationary Gaussian process, and then the second order contributions that are superimposed on the Gaussian model are obtained. In a sense a double perturbation is made, "one of the differential equations about some equilibrium point and the other about the 'Gaussian' point".

The second order contribution to the spectrum is obtained from a convolution on the spectrum of the linear process. Both the sums and differences of the frequencies of the linear process enter, and the concept of phase speed becomes meaningless. Additional properties of this non-linear (and hence non-Gaussian) random model have still to be determined and thus this paper is only part one on the results for this particular model.

W. J. Pierson, Jr. (New York, N.Y.)

4675:

Parkhomovskii, S. I. Impulsive symmetrical cavitational flow past a grid of plates. J. Appl. Math. Mech. **22** (1958), 794–799 (565–568 Prikl. Mat. Meh.).

The author solves the free boundary problem described in the title, thus extending results of M. I. Gurevich [Akad. Nauk SSSR. Prikl. Mat. Meh. **16** (1952), 116–118; MR **13**, 877]. *G. Birkhoff* (Cambridge, Mass.)

4676:

Roseau, Maurice. Nouvelles recherches sur la théorie des ondes liquides de gravité en profondeur variable. J. Math. Pures Appl. (9) **38** (1959), 25–59.

Three-dimensional waves on a beach of uniform slope are treated when the variation along the beach is simple harmonic. The fluid occupies the sector $0<x<\infty$, $-x\tan\alpha<y<0$, and the governing equations are

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-k^2\right)\phi(x,y)=0$$

in the sector, $\partial\phi/\partial y-\phi=0$ on $y=0$, and $\sin\alpha(\partial\phi/\partial x)+\cos\alpha(\partial\phi/\partial y)=0$ on $y\cos\alpha+x\sin\alpha=0$. At infinity ϕ is to be bounded, at the origin ϕ is if possible to be bounded or logarithmically infinity. In his thesis [*Contribution à la théorie des ondes liquides de gravité en profondeur variable*, Publ. Sci. Tech. Ministère de l'Air, no. 275, Paris, 1952; MR **15**, 754] the author has shown that such solutions exist if $k<1$. In the present paper the frequency range $k>1$ is studied. The following results are obtained: (1) Suppose that $k=1/\sin n\alpha$, where $n\alpha<\frac{1}{2}\pi$ and n is an integer. If n is odd, then a bounded solution exists [in agreement with the reviewer's result, Proc. Roy. Soc. London Ser. A **214** (1952), 79–97; MR **14**, 326]. If n is even, a solution with a logarithmic singularity at the origin exists. (2) For all other values of $k>1$ there are two solutions which are infinite near the origin like

$r^{-n/\alpha}$. All these solutions tend to zero at infinity. The solutions are exhibited in closed form as contour integrals, the integrands of which satisfy certain functional equations and can be expressed by means of other contour integrals.

A more detailed study of the properties of these solutions is to be published elsewhere.

F. Ursell (Cambridge, England)

4677:

Hyers, D. H.; and Ferling, J. A. On the local uniqueness problem for periodic surface waves of permanent type in a channel of infinite depth. Math. Mag. **31** (1957/58), 61–74.

The authors consider the uniqueness problem for periodic channel flows of permanent type in an infinitely deep channel, for two dimensional, incompressible, non-viscous, irrotational flow. The existence of such flows was established by Levi-Civita in 1925 [Math. Ann. **93**, 264–314].

In terms of the wave length λ, acceleration of gravity g, and velocity of propagation c, a constant, the dimensionless parameter $\gamma = g\lambda/2\pi c^2$ is obtained. The local uniqueness theorems obtained for such flows are given for γ in the neighborhood of each positive integer n, which, in turn, can be related to the case for which γ is in a neighborhood of $n=1$. The method of proof involves the study of an integral equation arising in an equivalent formulation to Levi-Civita's. A typical result is the theorem: There exist positive numbers δ and ε such that, for every value of the wave length λ and propagation constant c satisfying $1-\gamma<\delta$, there exists a unique non-trivial wave of permanent type, whose velocity vector (u, v) satisfies $u^2+v^2<\varepsilon^2$. If $0\leq\gamma-1<\delta$, only the trivial solution $u=v=0$ exists.

G. E. Latta (Stanford, Calif.)

4678:

Fjeldstad, Jonas Ekman. The free oscillations of water in an elliptic-parabolic basin. Arch. Math. Naturvid. **54** (1958), 45–56.

The law of depth is

$$h = h_0\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)$$

and the shallow water theory without friction is used. The effect of the earth's rotation is included. The author has succeeded in finding simple solutions in the form of polynomials in x and y. No comparison with measurement is made.

F. Ursell (Cambridge, England)

4679:

Davies, T. V. On the forced motion due to heating of a deep rotating liquid in an annulus. J. Fluid Mech. **5** (1959), 593–621.

This interesting paper gives exact solutions for the jet stream flow of a liquid in a heated rotated annulus. The partial differential equations are reduced to ordinary (non-linear) differential equations and these are solved for different assumptions about the momentum transport in the model—the arbitrariness is probably due to the neglect of viscosity. The solutions so obtained are shown to exist only for certain ranges of the Rossby number and the angular velocity of the system. When these results are combined with an earlier linearized calculation by the author [Philos. Trans. Roy. Soc. London Ser. A **249** (1956), 27–64; MR **17**, 1149] a stability diagram is obtained which compares qualitatively with experimental data.

M. H. Rogers (Bristol)

4680:

Ryzhov, O. S. On flows in the region of the transition surface. J. Appl. Math. Mech. **22** (1958), 608–621 (433–443 Prikl. Mat. Meh.).

Flow through an axially symmetric Laval nozzle is investigated in the neighborhood of the intersection of the axis of symmetry with a sonic line. It is assumed that the motion is irrotational, and a transonic approximation is used. The hodograph transformation is avoided, and the analysis depends instead on a Cauchy problem with data assigned along the axis. This leads to a similarity solution of the transonic equations permitting a comparison between the plane and axially symmetric cases, which have some, but not all, features in common.

P. R. Garabedian (New York, N.Y.)

4681:

Arynov, A. Gas flow from a vessel with plane walls forming a small angle $2\theta_0$. Soviet Physics. Dokl. **123** (3) (1958), 1106–1109 (43–46 Dokl. Akad. Nauk SSSR).

4682:

Gross, W. A. A gas film lubrication study. I. Some theoretical analyses of slider bearings. IBM J. Res. Develop. **3** (1959), 237–255.

The Reynolds equation of gas lubrication for slider bearings is solved on a digital computer. This differs from the classical solution that the density-pressure relation is represented by a polytropic expression. Numerical results of isothermal case are given in chart [see following reviews].

L. N. Tao (Chicago, Ill.)

4683:

Michael, W. A. A gas film lubrication study. II. Numerical solution of the Reynolds equation for finite slider bearings. IBM J. Res. Develop. **3** (1959), 256–259.

This part gives the difference equation which corresponds to the Reynolds differential equation of gas lubrication [see preceding and following review].

L. N. Tao (Chicago, Ill.)

4684:

Brunner, R. K.; Harker, J. M.; Haughton, K. E.; and Osterlund, A. G. A gas film lubrication study. III. Experimental investigation of pivoted slider bearings. IBM J. Res. Develop. **3** (1959), 260–274.

This part reports the result of an experimental study of gas lubricated slider bearings. It is compared to the computer solution of part I [see preceding reviews]. Best agreement occurs when the density-pressure relation is assumed linear (isothermal).

L. N. Tao (Chicago, Ill.)

4685:

Tao, L. N. On journal bearings of finite length with variable viscosity. J. Appl. Mech. **26** (1959), 179–183.

This paper gives an exact solution to the Reynolds lubrication equation for a journal bearing of finite length in the case that the viscosity $\mu=\mu_0 \exp(bp)$ where p is the pressure and b is a constant. The solution is expressed in terms of Heun functions. Upper and lower bound are found for the components of the load capacity along and perpendicular to the line joining the centers of the two cylinders. From these results it is deduced that the line of centers is not perpendicular to the load vector—which agrees with experiment. *R. C. DiPrima* (Troy, N.Y.)

4686:

Arkhipov, V. N. The formation of streaming fluctuations behind a solid obstacle. Soviet Physics. Dokl. **123** (3) (1958), 1117–1120 (620–622 Dokl. Akad. Nauk SSSR).

4687:

Watson, J. The two-dimensional laminar flow near the stagnation point of a cylinder which has an arbitrary transverse motion. Quart. J. Mech. Appl. Math. **12** (1959), 175–190.

The case of a two dimensional flow against an infinite flat plate normal to the stream and performing arbitrary transverse motion is considered. Expansions for the velocity distribution for small and large time are found in terms of the velocity of the wall by the use of Laplace transform techniques. Computations are carried out for the particular case of a wall set impulsively in uniform motion. An appropriate solution is also found by the use of a Pohlhausen type of method. If necessary the approximate solution can be used as a guide for linking the solutions for small and large time together.

R. C. DiPrima (Troy, N.Y.)

4688:

Lemechov, E. E. Steady turbulent motion of gas in round pipes. II. Vestnik Leningrad. Univ. **14** (1959), no. 7, 88–104. (Russian. English summary)

The article deals with turbulent motion of gas in a round pipe, the ratio T_w/T_0 of the temperatures of gas on the wall and on the axis of the pipe being constant. This special case is used as a basis for approximate solution of the problem in general. Besides, the author investigates the influence of the Reynolds and Prandtl numbers and of the parameter T_w/T_0 on the ratios of mean values of hydrodynamical elements to their extremal values, as well as on the corrections for kinetic energy and momentum.

Author's summary

4689:

Bourne, D. E.; Davies, D. R.; and Wardle, S. A further note on the calculation of heat transfer through the axisymmetrical laminar boundary layer on a circular cylinder. Quart. J. Mech. Appl. Math. **12** (1959), 257–260.

Seban et Bond ont donné une solution exacte de l'équation de la température sous la forme d'une des séries en $h=vx/Ua^2$ pour $h \leqq 0{,}04$ (U la vitesse uniforme du courant, x la distance en aval du courant, v viscosité cinématique et a le rayon du cylindre). D'autre part les auteurs ont obtenu une solution asymptotique valable pour des grandes valeurs de h ($h \geqq 100$).

Pour les valeurs intermédiaires les auteurs ont employé une méthode d'approximation basée sur le cas du transfert de la chaleur d'une plaque plate en introduisant quelques modifications [Bourne et Davies, même J. **11** (1958), 52–66; MR **20** #4991].

Dans cette note les auteurs complètent leurs calculs en utilisant les approximations de Karman-Polhausen d'après la méthode décrite par Glauert et Lighthill [Proc. Roy. Soc. London Ser. A **230** (1955), 188–203; MR **16**, 1171].

M. Kiveliovitch (Paris)

4690:

Merk, H. J. Mass transfer in the laminar boundary layer along a flat plate calculated by means of the integral method. Appl. Sci. Res. A **8** (1959), 261–277.

"The approximation method propounded by von Karman and Pohlhausen for hydrodynamic boundary layers has been applied to the calculation of the mass transfer in the laminar boundary layer along a flat plate without longitudinal pressure gradient. The velocity and concentration profiles are represented by polynomials of the 4th degree in terms of the distance from the wall. It is shown that for mass transfer this method leads to a rather complicated system of formulae. The calculations based upon this method are performed for several values of the Schmidt number Sc, viz. for $Sc=0.1$, 0.6, 1, 2, 5, 10, 50 and 100. The general trend of the results is in accordance with the exact theory. However, it appears that the numerical work involved is extensive and its accuracy disappointing. Furthermore, for Schmidt numbers lower than 1 the approximation method fails at certain negative values of the transfer parameter B." (Author's summary) *R. C. DiPrima* (Troy, N.Y.)

4691:

Squire, William. A note on the Blasius equation with three-point boundary conditions. J. Aero/Space Sci. **26** (1959), 678–679.

4692:

Williams, James C., III. The compressible laminar boundary layer with arbitrary pressure gradient and wall temperature distribution. J. Aero/Space Sci. **26** (1959), 677–678.

4693:

Rumyantsev, V. V. On stability of equilibria of a rigid body with liquid-filled cavities. Soviet Physics. Dokl. **124** (4) (1959), 46–49 (291–294 Dokl. Akad. Nauk SSSR).

Lagrange's stability theorem is shown to hold for a rigid body with cavities not completely filled with a liquid. Lagrange's theorem states that, if an equilibrium position is an isolated minimum of the potential energy, then the equilibrium is stable. Under the assumptions that the liquid is a homogeneous incompressible ideal fluid, that the forces are conservative, etc., there is an energy integral and the proof is elementary. Stability of the body and the fluid surface is in the sense of Liapunov and is carefully explained. It seems that in fluid mechanics Liapunov's concept of the stability of an equilibrium form of a fluid is not well known and also that previous discussions of this problem dealt only with the linear approxi-

mation. The author points out that the theorem is easily seen to be true for viscous fluids.

J. P. LaSalle (Baltimore, Md.)

4694:

Stewartson, K. On the stability of a spinning top containing liquid. J. Fluid Mech. **5** (1959), 577–592.

"The stability of a heavy top, containing a cylindrical cavity partly full of liquid, for small displacements from the sleeping position is studied. It is shown theoretically that instability can occur when any one of the periods of free oscillation of the liquid, which are doubly infinite in number, is sufficiently near to the period of nutation of the empty top. In experiments carried out by G. N. Ward (described in an Appendix), only the two principal instabilities could be distinguished." (From the author's summary) *D. W. Dunn* (Ottawa, Ont.)

4695:

Veronis, G. Cellular convection with finite amplitude in a rotating fluid. J. Fluid Mech. **5** (1959), 401–435.

A comprehensive theoretical investigation is made of the various effects of rotation upon the cellular convection which sets in when a fluid is heated uniformly from below and cooled from above. For simplicity the horizontal top and bottom surfaces of the cells are assumed to be free.

First, the linearized stability equations are used to review the dependence of the critical Rayleigh number for steady convective instability upon the Taylor number which measures the importance of rotation. The distorting effects of rotation upon cell shape are explained. It is shown that particles near the centres and corners of cells with originally square or hexagonal cross-section will trace out conical helices as they rise and descend; however, the effective wave number of the distorted cells remains unaltered. A further complication is that for sufficiently small Prandtl numbers and moderate to large Taylor numbers "overstability" will occur; i.e., the amplitudes of the cellular motions will fluctuate periodically with time.

The difficult non-linear problem of cells with finite circulations is next taken up and discussed as a perturbation of the linear solution for Rayleigh numbers neighbouring the critical value associated with infinitesimal disturbances. The condition that the perturbations be spatially periodic gives the amplitude of the motion in a manner somewhat analogous to that used in the Poincaré perturbation theory of limit cycles. One striking result already noted by Chandrasekhar is that, for small enough Prandtl numbers, finite cellular motions can be self-maintained in the presence of rotation for Rayleigh numbers less than the critical required for infinitesimal instability.

Conditions for finite amplitude overstability are also investigated and are found, for sufficiently low Prandtl number, to be favoured by increasing Taylor number. The actual cellular pattern which sets in for given values of the parameters, be it steady or overstable, may be determined by finding the mode with the maximum heat transfer. Predictions are made for mercury and air.

A. F. Pillow (Toronto, Ont.)

4696:

★Hinze, J. O. Turbulence: An introduction to its mechanism and theory. McGraw-Hill Series in Mechanical Engineering. McGraw-Hill Book Co., Inc., New York-Toronto-London, 1959. ix+586 pp. $15.00.

The purpose of this book is to present current knowledge and theories of turbulent flow in a way that will prepare a reader for study of more specialized work or for theoretical analysis of problems arising in chemical or mechanical engineering. The result is the first comprehensive account to appear since the publication of "Modern developments in fluid mechanics" over twenty years ago, and on this ground alone it must be recommended to anyone interested in the nature and properties of turbulent motion. The scope may be inferred from a brief list of contents:

Chapter 1. "General introduction and concepts". Equations of turbulent motion and of scalar transport; velocity correlations; macro- and microscales; Lagrangian description of diffusion; energy spectrum and energy equation. Chapter 2. "Principles of methods and techniques in the measurement of turbulent flows". Theory and use of the hot-wire anemometer; other techniques for measuring mean and fluctuation quantities. Chapter 3. "Isotropic turbulence". Equation for double-velocity correlation tensor; decay of isotropic turbulence; Kolmogoroff theory of equilibrium range of eddies; fluctuations of a convected scalar; pressure fluctuations. Chapter 4. "Non-isotropic turbulence". Statistical theory of homogeneous flow with uniform shear. Chapter 5. "Transport processes in turbulent flows". Mixing-length and other phenomenological theories; Reynolds analogy; Eulerian and Lagrangian description of the diffusion of fluid particles and of convected scalars; Diffusion of discrete particles. Chapter 6. "Non-isotropic free turbulence". Approximate forms of momentum and energy equations; mean flow in turbulent wakes and jets, comparison with theory; structure of free turbulent flow. Chapter 7. "Non-isotropic 'wall' turbulence". Approximate forms of momentum and energy equations; turbulent boundary layer on a flat plate; turbulent flow in pipes and channels; structure of pipe flow.

Although a considerable part of the book is concerned with theoretical work, a knowledge of advanced parts of mathematics is not necessary and, with the help of a short appendix dealing with the elementary theory of Cartesian tensors, all the work should be accessible to a graduate in physics or engineering.

The statistical theory of homogeneous turbulence and related topics, here treated in considerable detail, has attracted much attention in recent years but the gap between this theory and the phenomenological theories of shear flow is still wide, and a reading of this book makes this very clear. Professor Hinze makes a considerable effort in the sections on shear flow to bring out the assumptions underlying these theories and systematically compares their predictions with the experimental observations. If the impression left by these sections is one of confusion, the fault lies with the subject matter and not with the exposition which is very full and accurate. A very valuable feature is an extensive account of the theory and use of the hot-wire anemometer which is the basic instrument for the detailed study of turbulent flows. There are a few errors but only two seem to deserve mention. On page 36, the author follows tradition by equating the Taylor micro-scale, λ_g, and the size of the eddies responsible for viscous dissipation, although both experiment and the theory of universal equilibrium show

their size to be of order $\lambda_g Re_\lambda^{-1/2}$ (p. 184 et seq.). The second mistake occurs in the discussion of orders of magnitude in a turbulent boundary layer (p. 456), in which it is assumed that the scale of variation of mean velocity is of the same order as the mean velocity itself. This is true close to the viscous sub-layer, but everywhere else the scale of variation is comparable with the turbulent velocities and is an order of magnitude less than the absolute velocity. Neglect of this possibility leads to the conclusion that the shear coefficient, $R_{12} = -\overline{\mu_1\mu_2}/\overline{\mu_1^2}$, is small except close to the wall. In fact, as figure 7.15 shows, it is large and very nearly constant over the whole layer. *A. A. Townsend* (Cambridge, England)

4697:

Kraichnan, Robert H. The structure of isotropic turbulence at very high Reynolds numbers. J. Fluid Mech. 5 (1959), 497–543.

This long paper contains an account of a recent theory of homogeneous turbulence based on the assumption that statistical dependence between distinct Fourier components of the velocity field is as small as the equations of motion allow. This account, which differs slightly from the first form of the theory [Kraichnan, Phys. Rev. (2) **109** (1958), 1407–1422; MR **19**, 1221], considers in detail the nature of energy transfer between Fourier components of wave-number large compared with the wave-numbers of the energy-containing eddies, and derives expressions for the energy spectrum, time-correlations and skewness factors in this range. Like most work on the fundamental theory of turbulence, the paper requires careful reading with intervals for thought but it is an important piece of work that should be read by everyone with a serious interest in the subject.

The central idea is of weak dependence, that statistical dependence between distinct Fourier components approaches zero as the cubical volume used for the Fourier representation becomes infinite. Previous theories have considered infinite volumes and assume the dependence to be identically zero. The small dependence between groups of components is assumed to be induced by the non-linear terms in the equation of motion alone, so that the dependence can be calculated from a knowledge of the non-linear terms and the infinitesimal response tensor, defined so that the change in the Fourier component $u_i(\mathbf{k})$ due to a solenoidal body force $\xi(\mathbf{k}, t)$ introduced at $t = t_0$ is

$$\delta u_i(\mathbf{k}) = \int_{t_0}^{t} \zeta_{ij}(\mathbf{k}; t, t')\xi_j(\mathbf{k}, t')dt'. \tag{1}$$

The most effective of the non-linear terms leading to statistical dependence between components of wave-numbers $\mathbf{k}$, $\mathbf{p}$, $\mathbf{q}$ are the direct interaction terms,

$$b_i(\mathbf{k}) = -iP_{ijm}(\mathbf{k})u_j(-\mathbf{p})u_m(-\mathbf{q}),$$

etc., and the use of these terms to the exclusion of all others amounts to using the first term in a new kind of perturbation expansion of the equations. This approximation leads to definite relations between mean values of second and third order moments of the Fourier components and the mean impulse response tensor $\langle\zeta(k, \tau)\rangle$. Applied to stationary, isotropic turbulence generated by isotropic, stationary, solenoidal body forces, a closed system of equations can be derived which determines all these quantities in terms of the body forces. For wave-numbers much larger than the characteristic wave-number of the energy-containing eddies, k_0, these functions are independent of the body forces and are determined by k_0, v_0 (the r.m.s. velocity fluctuation) and the viscosity. In particular, the three-dimensional spectrum function is of the form

$$E(k) = \nu v_0(k/k_d)^{-3/2}f(k/k_d), \tag{2}$$

where $k_d = R_0^{3/2}k_0$, $R_0 = v_0k_0^{-1}/\nu$ and $f(0) \neq 0$, and the skewness factor of the longitudinal velocity derivative is not constant but proportional to $R_0^{-1/4}$. Both these results are at variance with the Kolmogoroff theory of local similarity, and the relation of the theory to this and other theoretical work is discussed in detail. The author considers that these new predictions are in slightly better agreement with the available experimental evidence than are the similarity predictions.

A. A. Townsend (Cambridge, England)

4698:

Tien, C. L. On the eddy diffusivities for momentum and heat. Appl. Sci. Res. A 8 (1959), 345–348.

A "proof" is presented that the eddy diffusivities in turbulent flow for momentum and for heat are identical. It depends on assuming that a fluid particle diffusing across the mean flow retains its original mean velocity or mean temperature until the Lagrangian correlation for transverse velocity fluctuations becomes negligible. Even if the viscosity is negligible, turbulent pressure gradients will cause the particle to "forget" its initial longitudinal velocity as rapidly as it forgets its transverse velocity. A semi-empirical expression for the variation of eddy diffusivity in a pipe is given, based on proportionality of the Eulerian and Lagrangian scales of turbulence.

A. A. Townsend (Cambridge, England)

4699:

Haskind, M. D. On transport theory. Motion of a heavy particle in a turbulent flow. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1956**, no. 11, 28–39. (Russian)

Dans le travail sur une particule pesante dans un courant turbulent de M. Velikanov [Dokl. Akad. Nauk SSSR **85** (1952), 555–558] l'auteur essaie de démontrer que les vitesses longitudinales des particules et du courant sont les mêmes tandis que les vitesses verticales diffèrent d'une grandeur égale à la vitesse de la particule libre dans un liquide immobile. En reprenant le problème dans toute sa généralité, et en faisant le bilan de toutes les forces qui agissent sur la particule M. Velikanov n'a pas pris en considération l'effet de l'inertie du liquide; d'autre part l'hypothèse d'une loi linéaire de résistance n'est pas toujours valable et dépend de l'ordre de grandeur des nombres de Reynolds, la linéarité peut être admise seulement pour des petits nombres de Reynolds.

On arrive à cette conclusion que c'est seulement lorsque les fréquences des pulsations du courant turbulent sont faibles, que les vitesses longitudinales du courant et des particules sont égales. Lorsque les fréquences deviennent plus fortes cette conclusion n'est plus valable.

L'auteur étudie ensuite le mouvement des particules pour une loi quadratique de résistance ainsi que les propriétés qu'emportent les particules dans leur mouvement. *M. Kiveliovitch* (Paris)

4700:

Dumitrescu, Lucian. Sur le transfert de chaleur dans l'écoulement en régime de molécules libres. Acad. R. P. Romîne. Stud. Cerc. Mec. Apl. **10** (1959), 379–392. (Romanian. Russian and French summaries)

4701:

v. Krzywoblocki, M. Z. On the mathematical formulation of the motion of an inviscid compressible fluid in axial compressors. I, II. Comment. Math. Univ. St. Paul **7** (1959), 35–55, 77–97.

The equations of motion of an inviscid compressible fluid are found referred to general non-orthogonal, curvilinear, coordinate systems. A few comments on possible methods of solution are also given.

K. Stewartson (Durham)

4702:

Power, G.; and Smith, P. Approximate subsonic gas flows under assigned boundary conditions. Appl. Sci. Res. A **8** (1959), 349–356.

Cet article est consacré aux écoulements d'un fluide obéissant à la loi d'état de Chaplygin—la pression est une fonction linéaire du volume spécifique. La théorie de ces écoulements est identique à celle des surfaces minima. Certaines formules de cette théorie sont interprétées en termes d'écoulements. *P. Germain* (Paris)

4703:

Power, G.; and Smith, P. Application of a reciprocal property to subsonic flow. Appl. Sci. Res. A **8** (1959), 386–392.

Les auteurs énoncent un principe de correspondance entre écoulements de fluides compressibles et appliquent ce principe à la recherche d'une bonne approximation de Chaplygin pour étudier l'écoulement autour du cercle lorsque le nombre de Mach à l'infini est 0.4.

P. Germain (Paris)

4704:

Spreiter, John R. Aerodynamics of wings and bodies at transonic speeds. J. Aero/Space Sci. **26** (1959), 465–486, 517.

This is an admirably clear and comprehensive survey of transonic small-disturbance theory applied to thin wings and slender bodies. The various aspects of the theory are illuminated at each stage by critical comparison with experimental results and more elaborate theories.

An introduction to the small-disturbance equations and their similitude properties is followed by a discussion of Oswatitsch' equivalence rule, the transonic area rule, and their interrelation. Results of two classical methods are considered briefly: the hodograph transformation for plane flow, and the method of successive approximations based upon linearization. It is suggested that the series resulting from the latter can be recast into a second form that, together with the first, brackets the solution. Next, two recent approximate methods of solving the nonlinear equations are discussed. The first involves transforming the partial differential equation to an integral equation, assuming a certain "profile" for the cross-stream variation of the unknown velocity in the nonlinear term, and solving the resulting single integral equation numerically. The second, the method of "local linearization", involves temporarily linearizing the equation (approximating the nonlinear term $\phi_x\phi_{xx}$ by $\lambda\phi_{xx}$ for locally subsonic or supersonic flows, but by $\lambda\phi_x$ in nearly sonic regions), solving the resulting Laplace, wave, or heat equation to obtain a representation formula for the velocity at the body surface, and at that stage abandoning the linearization (restoring ϕ_x or ϕ_{xx} in place of λ). This yields an ordinary nonlinear first-order differential equation that can be integrated in closed terms. Although the mathematical justification of this procedure remains obscure, it gives very simple solutions that are, for a wide variety of bodies, in remarkable accord with experiments and more exact theories. A further approximation is suggested that extends to transonic flow Whitham's method for rendering the distant flow field uniformly valid. Finally, several provocative views are advanced regarding wind-tunnel wall interference at sonic speed. Strong support is given to the view that the best way of simulating a free-stream Mach number of unity is to abandon slotted and porous walls, and simply choke an old-fashioned solid-wall wind-tunnel.

M. D. Van Dyke (Stanford, Calif.)

4705:

Sharp, A. W. The supersonic flow past a leading edge separation bubble. J. Fluid Mech. **5** (1959), 445–459. (1 plate)

Two-dimensional symmetrical supersonic flow past a square-nosed flat plate involves a detached bow shock wave with a resulting region of mixed flow. The boundary layer separates at the corner and reattaches downstream to form the so-called separation bubble. From a schlieren photograph and a static-pressure distribution the author determined experimentally shock shape, bubble shape, and velocity on the bubble. Then he calculated in two stages by the method of characteristics the flow in part of the supersonic region and determined that portion (about half) of the sonic line adjacent to the corner. In the first stage the flow was considered isentropic. From approximate values of mass flow obtained from the first stage, and by means of the experimental shock shape he inserted the entropy variations into the characteristic equations to calculate a second approximation. In the immediate neighborhood of the sonic line the inclinations of characteristic segments that intersect on the sonic line were determined by an averaging process that exploits the fact that here the characteristics are approximately semi-cubical parabolae. *J. H. Giese* (Aberdeen, Md.)

4706:

Nocilla, Silvio. Flussi transonici attorno a profili alari simmetrici con onda d'urto attaccata ($M_\infty < 1$). II. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **93** (1958/59), 124–148.

4707:

Cheng, Hsien K. Similitude of hypersonic real-gas flows over slender bodies with blunted noses. J. Aero/Space Sci. **26** (1959), 575–585.

Hypersonic small-disturbance theory is generalized to apply to real gases in thermodynamic equilibrium, and to plane or axisymmetric bodies that, though otherwise

slender, have slightly blunted leading edges. Similitude is shown to exist for related families of bodies if not only the classical hypersonic similarity parameter but also the free-stream atmosphere and a nose-drag parameter are the same. The similarity rules are illustrated by application to previous numerical solutions for blunt-nosed plates in ideal gas and for wedges in argon-free air.

M. D. Van Dyke (Stanford, Calif.)

4708:

Kogan, Abraham. On 2-dimensional shock-waves for near-detachment flow. Bull. Res. Council Israel. Sect. F **7** (1957/58), 123–128.

Crocco's stream function and the abscissa are taken as independent variables in an equation relating velocity and entropy. Near the tip of a sharp-nosed airfoil with nearly detached bow wave, derivatives with respect to the second variable are shown to be negligible. Integrating the resulting ordinary differential equation gives approximate relations for surface pressure and temperature.

M. D. Van Dyke (Stanford, Calif.)

4709:

Želudev, P. I. Supersonic flow around plane quasi-triangular wings of small span. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk. Meh. Mašinostr. **1959**, no. 3, 202–203. (Russian)

A superficial attempt is made to iterate on Jones' slender-wing theory to find a second approximation. The result is worthless, among other reasons, because the transonic term is neglected and no outer boundary condition is imposed. *M. D. Van Dyke* (Stanford, Calif.)

4710:

Čurikov, F. S. On one form of the equations of supersonic flow of a gas. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk. Meh. Mašinostr. **1959**, no. 3, 204–207. (Russian)

Many supersonic flow problems are governed by a pair of first-order quasi-linear partial differential equations of hyperbolic type for two functions of two independent variables. The author investigates conditions under which these may be reduced to the single "compact" equation $w_{\xi\eta}=a(\xi, \eta)w$. This is shown to be possible for one-dimensional unsteady isentropic and two-dimensional steady irrotational supersonic flows, and for two different formulations of slightly supersonic steady flow.

M. D. Van Dyke (Stanford, Calif.)

4711:

Shmyglevskii, Iu. D. On some properties of axially symmetric supersonic gas flows. Soviet Physics. Dokl. **122** (3) (1958), 928–930 (782–784 Dokl. Akad. Nauk SSSR).

Part of the wave front analysis of axially symmetrical supersonic flow of R. E. Meyer [Quart. J. Mech. Appl. Math. **1** (1948), 451–469; MR **10**, 492] is extended to flow with entropy variation. Given that the curvature of the body contour decreases in the stream direction at a point A on the contour, the curvature of the nose shock is shown to decrease with distance from the body at the point where the Mach line issuing from A meets the shock.

R. E. Meyer (Providence, R.I.)

4712:

Tan, H. S. On a special Bolza variational problem, and the minimization of superaerodynamic hypersonic nose drag. Quart. Appl. Math. **17** (1959), 311–314.

This paper is concerned with the problem of obtaining an optimum nose curve that minimizes the superaerodynamic hypersonic drag of an axial symmetric body. This leads to a variational problem of the Bolza type without differential side conditions. As is to be expected singularities arise leading to restrictions on the slope y'. The problem formulated can be solved explicitly. The author discusses the properties of the solution. A minor misprint appears in the integrand F (y' for y'^2).

M. R. Hestenes (Los Angeles, Calif.)

4713:

Carafoli, Elie; et Mateescu, Dan. Écoulement supersonique autour du système portant aile-fuselage conique. Acad. R. P. Romîne. Stud. Cerc. Mec. Apl. **10** (1959), 335–350. (Romanian. Russian and French summaries)

4714:

Levey, H. C. The thickness of cylindrical shocks and the PLK method. Quart. Appl. Math. **17** (1959), 77–93.

This is a further contribution to the author's earlier work on the steady radial flow of a viscous heat conducting gas [H. C. Levey, same Quart. **7** (1954), 25–48; MR **16**, 190]. Source flow was considered in the earlier paper and it is now shown that similar results are obtained for sink flow. In particular a shock of thickness $R_e^{-1}S^{-1}\log(R_eS^2)$ is found, where R_e is the Reynolds number based on mass flux and viscosity, and S is the shock strength. The shock structure agrees with the usual plane shock theory; the additional logarithmic factor in the shock thickness seems to result from a special definition of thickness. In discussing sink flow, Wu [T. Y. Wu, ibid. **13** (1956), 393–418; MR **17**, 552] had found similarity solutions with shocks of thickness $R_e^{-2/3}$, and had used a Lighthill type of expansion procedure to deduce that these included all solutions. The author notes that this particular expansion procedure is not appropriate here and that the conclusion is wrong. Wu's similarity solutions cover the special cases when the shock is close to the sonic line (a singularity of the inviscid solution) and $S\propto R_e^{-1/3}$. Boundary layer techniques are appropriate for an analytic treatment of the general case and they have been applied by Sakurai [A. Sakurai, Quart. J. Mech. Appl. Math. **9** (1958), 274–289; MR **20** #547].

G. B. Whitham (Cambridge, Mass.)

4715:

Rosciszewski, Jan. Influence of shock-wave attenuation by boundary-layer growth on contact-surface motion in a shock tube. J. Aero/Space Sci. **26** (1959), 388–389.

Use is made of a perturbation theory of characteristics of anisentropic one-dimensional compressible flow to study the motion of a contact surface behind a shock wave which is being attenuated in its motion along a duct by the formation of a boundary layer behind it. A rather brief and obscure argument is given to show that a range of shock deceleration values exists for which the contact surface will be accelerated as has been observed by Glass and Patterson. *A. F. Pillow* (Toronto, Ont.)

4716:

★**Murthy, S. N. B. On some one dimensional expansion processes with shock waves.** Proceedings of the Third Congress on Theoretical and Applied Mechanics, Bangalore, December 24–27, 1957, pp. 235–242. Indian Society of Theoretical and Applied Mechanics, Indian Institute of Technology, Kharagpur, 1958. xi+362 pp.

Some very particular solutions are obtained for the equations of conservation of mass and momentum in one-dimensional unsteady flow. Thermodynamic laws are not prescribed a priori. *D. C. Pack* (Glasgow)

4717:

Ehlers, F. Edward; and Shoemaker, E. M. A linearized analysis of the forces exerted on a rigid wing by a shock wave. J. Aero/Space Sci. **26** (1959), 75–80, 107.

Using A. Busemann's method of conical flow in the manner of Keller and Blank [Comm. Pure and Appl. Math. **4** (1951), 75–94; MR **13**, 304] there are obtained explicitly the pressure distributions produced by a plane weak shock incident upon a moving half plane or stationary wedge with edge parallel to the incident front.

A. A. Blank (New York, N.Y.)

4718:

Collar, A. R. Iterative solutions of the equations for plane oblique shock waves. J. Roy. Aero. Soc. **63** (1959), 669–672.

4719:

Kogan, M. N. On the equations of motion of a rarefied gas. J. Appl. Math. Mech. **22** (1958), 597–607 (425–432 Prikl. Mat. Meh.).

After a brief review of the process by which the Navier-Stokes equations are obtained from the Maxwell-Boltzmann equation for relatively dense gases, the author undertakes an improvement for slightly rarefied gases. Previous efforts in this direction by Enskog, Burnett, and Grad are mentioned. Here the right-hand side of the Maxwell-Boltzmann equation is written approximately as the difference between distribution functions, on the basis of a physical argument and a numerical verification for three simple cases. Although this was previously used by Bhatnagar, Gross, and Crook [Phys. Rev. (2) **94** (1954), 511–525] the present author has supplied the justification and concludes that the approximation is unsuitable for the particular case to which the earlier authors applied it. The equation is integrated and leads to a distribution function completely determined by the hydrodynamic quantities and their derivatives. All terms of Enskog's series are determined. For some cases in which the exact solution is known, the present method gives results correct in form but with slightly different constants. The paper closes with a remark concerning highly rarefied gases, wherein collisions are unimportant.

W. R. Sears (Ithaca, N.Y.)

4720:

★**Herzfeld, Karl F.; and Litovitz, Theodore A. Absorption and dispersion of ultrasonic waves.** Pure and Applied Physics, Vol. 7. Academic Press, New York-London. xviii+535 pp. $14.50.

Despite the generality of its title, this book is limited to phenomena in the gaseous and liquid states, so that a less extensive title would have been more appropriate.

The text is divided into three roughly equal parts: A—the general theory of relaxation in fluids; B—gases; and C—liquids.

Part A begins from the Navier-Stokes equation of hydrodynamics, introduces general relaxation considerations, and applies these to sound waves via the several approaches of thermodynamics, kinetic theory and statistics. This part constitutes the mathematical formulation of the macroscopic aspects of the phenomena of absorption and dispersion, and is largely formal in character. If much of the treatment is but a repeat of original works of many researchers, including the authors themselves, or of the reviews of other authors, it is nevertheless a prime asset of the book that all these developments are set down in one place, and with unified notation. It replaces, and greatly expands, the similar treatment by Markham, Beyer and Lindsay in 1951 [Rev. Mod. Phys. **23** (1951), 353–411; MR **13**, 599].

The parts devoted to gases and to liquids follow the same pattern: application of Part A to the particular state; a brief discussion of the appropriate methods of measurements; a summary of experimental results; and, finally, an attempt to survey the studies of the physical origins of the processes formally described in Part A. In many respects, those latter sections of Parts B and C are the most interesting portions of the book.

While the overall level of the text is high, and the mathematical treatment is sufficiently rigorous, at least for physicists, there are some weak points. The number of similar ways of treating relaxation being large, the reader in going through the book begins to feel that he has read the same chapter several times. While identical repetition does not occur, a more unified presentation, or one in which the intended direction was more clearly defined, would have improved the book.

The chapters on liquids and gases would have been aided by a few illustrations, and the extremely slight attention given to high intensity sound waves might better have been omitted or amplified.

Finally, the method of renumbering footnotes for each of the 118 sections of the text is a rather disconcerting one, especially on a page containing parts of two sections.

These complaints, however, should not detract from the solid contribution that the book makes to the field. Many analyses, from sources in several disciplines, have been brought together in one book. This volume will certainly serve as the future starting point for everyone beginning research in the field, as well as a source book for previous literature. *R. T. Beyer* (Providence, R.I.)

4721:

Prokof'ev, V. A. Weak waves in a compressible fluid taking radiation into account. Prikl. Mat. Meh. **21** (1957), 775–782. (Russian)

"Equations of radiation transport in a moving medium and hydromechanic equations taking heat flow due to radiation into account as well as the mechanical effect of radiation and internal energy are used to calculate the influence of radiation on the propagation of waves of infinitesimal amplitude. The hydrodynamic theory of the propagation of plane perturbations is analyzed using linearized equations. A characteristic (frequency) equation

is obtained which defines the damping and propagation rate of the forced and free modes in an equilibrium medium under very broad assumptions on the state of the medium and its optical properties. Examples are presented for certain scattering indices." (From the author's summary)

M. D. Friedman (Needham Heights, Mass.)

4722:

Weis, O. Über die Schallausbreitung in verlustbehafteten Medien mit komplexem Schub- und Kompressionsmodul. Acustica **9** (1959), 387–398. (English and French summaries)

"The wave propagation in an infinitely extended plate and in a liquid layer without shear tensions, having pressure release terminations, is investigated under the assumption of complex moduli. Besides results related to attenuation and phase velocity, conclusions are obtained with respect to the modes of propagation in a non-dissipative medium." *From the author's summary*

4723:

Lamb, George L., Jr. Diffraction of a plane sound wave by a semi-infinite thin elastic plate. J. Acoust. Soc. Amer. **31** (1959), 929–935.

The author applies the Wiener-Hopf technique in a new extension of half plane diffraction theory. In this case a thin semi-infinite elastic plate is the obstacle to an incident plane acoustic wave of wavelength large compared with the plate thickness. The conditions at the plate surface are that the normal velocity is continuous, and that the pressure discontinuity is related to the surface displacement by the fourth order thin plate differential equation. Although the results are quite general, within the restrictions already mentioned, easy approximation is only possible (i) when the plate density relative to the density of the surrounding medium is very large compared with the thickness/wavelength ratios, or (ii) when the ratio of the velocity of flexural waves in the plate to the sound velocity is very large compared to this same thickness/wavelength ratio.

{In the reviewer's opinion there are better methods for arriving at the functional equation which is solved by the Wiener-Hopf method than that of introducing Green's functions. Thus by taking transforms of the boundary conditions as well as of the differential equations the author's equation 11 may be obtained with much less labour [see e.g. B. Noble, *Methods based on the Wiener-Hopf technique for the solution of partial differential equations*, Pergamon, New York-London-Paris-Los Angeles, 1958; MR **21** #1505].}

V. M. Papadopoulos (Providence, R.I.)

4724:

Skalak, Richard; and Friedman, M. B. Reflection of an acoustic step wave from an elastic cylinder. J. Appl. Mech. **25** (1958), 103–108.

Following a modal analysis previously introduced by R. D. Mindlin and H. H. Bleich [same J. **20** (1953), 189–195; MR **15**, 485], a formal solution is obtained through standard Fourier transform techniques for the reflected and diffracted pressure fields at large distances from an elastic circularly cylindrical shell struck by a plane step (shock) pressure wave transversely to the cylinder axis. The usual acoustic approximations and linearizations apply. Specific numerical results are obtained for the first five modes for a rigid fixed cylinder and for a rigid neutrally buoyant cylinder. The pressures obtained in these cases are compared; in each case the reflected pulse is of essentially short duration (unit time is transit time of the original wave across the cylinder), the decay time being shorter for the floating cylinder.

P. Chiarulli (Chicago, Ill.)

4725:

Fraenkel, L. E. A cylindrical sound pulse in a rotating gas. J. Fluid Mech. **5** (1959), 637–649.

The propagation of weak shocks, or sound waves, into a perfect gas rotating at constant angular velocity is studied. The undisturbed pressure and density grow like $\exp(r^2)$, where r is a suitable dimensionless distance from the axis of symmetry of the motion. Sources on the axis generate waves that are shown to decay like $\exp(-r^2/2)$.

P. R. Garabedian (New York, N.Y.)

4726:

Heckl, M. Schallabstrahlung von Platten bei punktförmiger Anregung. Acustica **9** (1959), 371–380. (English and French summaries)

4727:

Gol'dberg, Z. A. Propagation of plane acoustic waves of finite amplitude in a viscous heat-conducting medium. Soviet Physics. Acoust. **5** (5) (1959), 117–119 (118–120 Akust. Ž.).

4728:

Lyamshev, L. M. A method for solving the problem of sound radiation by thin elastic shells and plates. Soviet Physics. Acoust. **5** (5) (1959), 122–124 (122–124 Akust. Ž.).

4729:

Blankfield, Judith; and McVittie, G. C. A method of solution of the equations of magnetohydrodynamics. Arch. Rational Mech. Anal. **2** (1958/59), 411–422.

The equations of classical magneto-hydrodynamics for an infinitely conducting medium are solved in terms of eight indeterminate functions. These are to be determined by the boundary conditions of the problem to be solved and turn out to be degenerate forms of the generalized potentials of relativity theory. The specific example which is worked out in detail refers to one-dimensional magneto-hydrodynamics in which the dependent variables involve the time and one linear coordinate only. The physical situation is that of a slab of gas, of infinite lateral extent, with parallel plane boundaries. It is shown that in adiabatic motion the slab can expand until it obtains a finite width and an equilibrium configuration. As the assumption of infinite conductivity is valid only for times short compared to the resistive relaxation time $t=l\sigma\mu$ the above solution is followed by a slower but infinite expansion.

W. P. Allis (Cambridge, Mass.)

4730:

Lenard, A.; and Bernstein, Ira B. Plasma oscillations

with diffusion in velocity space. Phys. Rev. (2) **112** (1958), 1456–1459.

The effect of small-angle collisions on longitudinal plasma oscillations is studied for a simplified model of a fully ionized gas. The analysis is based on a linearized relation representing a diffusion in velocity space which has the desired properties that the number of electrons is conserved and that the Maxwell distribution is the equilibrium state; namely,

$$(1)\qquad \frac{\partial f_1}{\partial t}+\mathbf{v}\cdot\frac{\partial f_1}{\partial \mathbf{r}}-\frac{e}{m}\mathbf{E}\cdot\frac{\partial f_0}{\partial \mathbf{v}}=\beta\frac{\partial}{\partial \mathbf{v}}\cdot\left[\mathbf{v}f_1+v_0^2\frac{\partial f_1}{\partial \mathbf{v}}\right].$$

Here $f=f_0+f_1$ denotes the joint distribution in position and velocity space divided by equilibrium density N, and $f_0=(2\pi v_0^2)^{-3/2}\exp[-\frac{1}{2}(v/v_0)^2]$; $\partial/\partial\mathbf{v}$ is the gradient in velocity space, e electron charge, m electron mass, $\mathbf{E}$ electric field, β an effective collision frequency. The perturbation f_1 is assumed to be very much smaller than f_0 and of the same order as $\mathbf{E}$. Equation (1) is solved with the Maxwell equations

$$(2)\qquad \frac{\partial}{\partial \mathbf{r}}\times\mathbf{E}=0;\quad \frac{\partial}{\partial \mathbf{r}}\cdot\mathbf{E}=-4\pi eN\int d^3v f_1.$$

The solution is a normal mode analysis; that is, the Laplace transform in time is inverted in terms of its poles. In the limit of weak collisions, $\beta\to 0$, there is recovered the result of Landau [Acad. Sci. USSR, J. Phys. **10** (1946), 25–34; MR **9**, 140]. *A. A. Blank* (New York, N.Y.)

4731:

Ludford, G. S. S. The propagation of small disturbances in hydromagnetics. J. Fluid Mech. **5** (1959), 387–400.

Plane wave solutions are obtained by linearization for a finitely conductive gas governed by the fluid equations

$$\frac{\partial \rho}{\partial t}+\operatorname{div}\rho\mathbf{v}=0,\quad \rho\frac{d\mathbf{v}}{dt}=-\operatorname{grad}p+\rho_e\mathbf{E}+\mu\mathbf{J}\times\mathbf{B}$$

with the thermodynamic relation

$$\rho T\frac{dS}{dt}=\frac{1}{\sigma}(\mathbf{J}-\rho_e\mathbf{v})^2$$

and Maxwell's equations

$$\mu\frac{\partial \mathbf{H}}{\partial t}=-\operatorname{curl}\mathbf{E},\qquad \rho_e=\varepsilon\operatorname{div}\mathbf{E},$$

$$\varepsilon\frac{\partial \mathbf{E}}{\partial t}+\mathbf{J}=-\operatorname{curl}\mathbf{H},\qquad \operatorname{div}\mathbf{H}=0$$

together with Ohm's law in the form

$$\frac{1}{\sigma}(\mathbf{J}-\rho_e\mathbf{v})=E+\mu\mathbf{v}\times\mathbf{H}.$$

The permeability μ, dielectric constant ε, and conductivity σ are assumed constant. The temperature T and specific entropy S are functions of pressure p and density ρ alone. The equations are linearized by considering a perturbation of the constant state $E=0$, $H=H_0$, $v=0$, $p=p_0$, $\rho=\rho_0$ and ignoring terms involving disturbances higher than the first order. (Since the fluid equations are given in Galilean invariant form and the Maxwell equations in Lorentz invariant form, the system of equations above lacks invariant properties. Coincidental to this particular perturbation scheme, however, it turns out that the linearized equations are Galilean invariant with $\mathbf{H}$ and $\mathbf{E}+\mu\mathbf{v}\times\mathbf{H}$ as the appropriate electromagnetic invariants.) A dispersion relation is obtained which splits into factors of fourth and fifth degree. The factorization is reflected in a splitting of the system of linearized equations. (It should be noted that a corresponding separation in the initial and boundary conditions is not generally possible.)

The problem of standing waves in a rectangular box with perfectly conducting walls is solved explicitly.

A. A. Blank (New York, N.Y.)

4732:

Taub, A. H. On circulation in relativistic hydrodynamics. Arch. Rational Mech. Anal. **3**, 312–324 (1959).

The relativistic analogue of the circulation in classical hydrodynamics is found. It is shown to vanish if and only if the flow is irrotational and isentropic; further, it is a constant of the motion if and only if the pressure is a function of density only. Finally, the effect of shocks and the relation between circulation and Bernoulli's theorem are discussed.

The argument is an elegant illustration of the methods and power of tensor calculus. *K. Stewartson* (Durham)

4733:

Vallée, Robert. Sur l'hydrodynamique en théorie de Jordan-Thiry. C. R. Acad. Sci. Paris **248** (1959), 1779–1781.

La nota ha per oggetto un tentativo di estendere alla teoria idrodinamica relativistica di Jordan-Thiry nella varietà V_5 lo schema della teoria della relatività generale riguardante la materia pura carica.

G. Lampariello (Rome)

4734:

Lauwerier, H. A. A diffusion problem with chemical reaction. Appl. Sci. Res. A **8** (1959), 366–376.

A fluid flows through a tube $0\leq r\leq R$, $z\geq 0$ with steady Poiseuille flow so that its velocity along the tube is proportional to R^2-r^2. A substance with concentration $C(r,z)$ diffuses in that fluid; it also disappears under chemical reaction at a rate proportional to $C(r,z)$. Its concentration is uniform at $z=0$, as the fluid enters the tube. Then $C(r,z)$ satisfies the conditions $D\nabla^2C-V(1-r^2/R^2)C_z-KC=0$ when $0\leq r<R$ and $z>0$, $C(r,0)=C_0$ and $C_r(R,z)=0$, where D, V and K are positive constants. The problem is simplified by assuming that diffusion in the z direction can be neglected; thus ∇^2C is replaced by $C_{rr}+r^{-1}C_r$. Then the author obtains a representation of C in the form of a series that involves orthogonal hypergeometric functions of r and exponential functions of z. For concentrations near the wall $r=R$, another formula for C is found by writing $y=R-r$ and approximating the partial differential equation by the equation $DC_{yy}-2VR^{-1}yC_z-KC=0$. Laplace transforms are used on this last approximation to get a simple series for $C(R,z)$. *R. V. Churchill* (Ann Arbor, Mich.)

OPTICS, ELECTROMAGNETIC THEORY, CIRCUITS

See also 4264, 4439, 4577, 4800, 4805, 4806.

4735:

Linfoot, E. H. Quality evaluations of optical systems. Opt. Acta **5** (1958), 1–14.

In recent years research workers in several fields of engineering and physics have been exploring the possibilities arising from the introduction of the techniques of information theory. One of the most promising opportunities which these techniques seem to offer to optics is in connection with the problem of image assessment. The present paper is a contribution in this direction.

The author points out that there are two distinct types of assessment, one in which a demand is made that the image shall resemble the object as closely as possible. The other that it shall give a maximum amount of information about the object, without regard to the amount of work required to extract the information. Several criteria of both types are considered and mathematical relations which exist between them are investigated. *E. Wolf* (Rochester, N.Y.)

4736:

de Witte, A. J. Equivalence of Huygens' principle and Fermat's principle in ray geometry. Amer. J. Phys. **27** (1959), 293–301.

In this paper the author has shown that the geometrical optics description of the propagation of light rays according to Fermat's principle is equivalent to the geometrical interpretation of Huygens principle (method of constructing wave fronts), or what one now would call the mathematical formulation of Huygens' principle from the Hamiltonian theory of geometrical optics. The analysis is limited to two dimensions, by considering a medium where the velocity is a function of position and direction. The path of the rays is given by the solution of the Lagrangian equation of the Fermat problem of least time. On the other hand, by considering the wave front associated with this problem, it is shown that the form of the velocity of propagation in the medium derived from the application of Huygens' method of construction leads to the same canonical integrand of the variation problem for the light rays. Finally the author gives a derivation of the laws of reflexion and refraction according to both methods.

{Reviewer's remarks: The formulation of the geometrical problem of propagation of light in non-homogeneous and anisotropic media according to Fermat's principle and the equivalent description by Huygens' method (construction of the wave fronts) is old and well known. However, it should be pointed out that the Hamiltonian formulation of Huygens' principle of constructing wave fronts is more general, since the latter describes the whole field of rays (two-parameter family of extremals) between any two wave fronts, whereas according to Fermat's principle one obtains solutions of individual rays. The particular case treated by the author in proving the equivalence of both methods is given in all its generality in books on partial differential equations and calculus of variations, where the extremals (light rays) of a variation problem are nothing else than the characteristics of Hamilton's (or in mechanics, Hamilton-Jacobi's) partial differential equation, or the so-called "eiconal" equation, or the bicharacteristics of the wave equation [Caratheodory, *Variationsrechnung und partielle Differentialgleichungen erster Ordnung*, Teubner, Berlin, 1935; Courant and Hilbert, *Methoden der mathematischen Physik*, I, II, Interscience, New York, 1943; MR **5**, 97; and also books on Optics, for instance, Caratheodory, *Geometrische Optik*, Springer, Berlin, 1937; R. K. Luneberg, *Mathematical theory of optics*, Brown University (1944), Advanced Instruction and Research in Mechanics, Providence, R.I., 1944; MR **6**, 107.] The proof of the equivalence of the two principles has been derived by many authors, including the reviewer, for optical, as well as electron optical, media [see Ph. Frank, Z. Phys. **80** (1933), 4–18; Niels Arley, Danske Vid. Selsk. Math.-Fys. Medd. **22** (1945), no. 8; and especially Luneberg in his New York University lecture notes on Propagation of Electromagnetic Waves (1947–48), where general anisotropic media are considered].} *N. Chako* (Flushing, N.Y.)

4737:

Colombo, Giuseppe. Sulle orbite stabili in un sincrotrone con tratti rettilinei. Ann. Mat. Pura Appl. (4) **46** (1958), 249–263.

Alcune ricerche fisiche concernenti le orbite stabili in un sincrotrone a tratti rettilinei hanno condotto ad un problema relativo ad un'equazione differenziale del 2° ordine del seguente tipo: $\ddot{y}+\varphi(y)y=p(t)$. Scopo della ricerca è di determinare, se esistono, le soluzioni della equazione che siano continue e limitate per tutti i valori di t da 0 a $+\infty$, particolarizzando le funzioni $\varphi(y)$ e $p(t)$. *G. Lampariello* (Rome)

4738:

Nilsson, Sven Gösta. The motion of electrons in two combined magnetic fields. Kungl. Tekn. Högsk. Handl. Stockholm. No. 72, 22 pp. (1953).

This paper deals with the focusing properties of a toroidal magnetic field in superposition with a constant axial magnetic field. Starting with the relativistic equation of motion of the electron (β-rays), a non-linear second order differential equation of the form

$$f+w^2f+bf^2+df^3+ef^4=c \tag{1}$$

has been derived by making the appropriate approximations. Here f is the radial displacement from the equilibrium orbit, w the angular frequency and b, d, etc., are constants depending on the ratio of the toroidal field B_θ to the axial field B_z. A system of non-homogeneous equations is obtained in the variable $y=qf$ when the frequency of the solution and y are expanded in terms of the parameter q ($q<1$). A periodic solution $y<1$, up to third order in q, is obtained which satisfies initial conditions. The frequency is also given up to the same order of the parameter. A third order approximation of the radial distance in terms of the radial and axial initial directions of the displacement of the electrons has been derived for the case of vanishing toroidal field. From these solutions the focusing properties (resolving power, dispersion) of such an arrangement (spectrometer) are derived for a small two-dimensional source placed symmetrically with respect to the toroidal axis, for half and a complete period of oscillation. For large toroidal fields in comparison to the axial field, the focusing properties

approach that of the straight solenoid. Even when the ratio of the toroidal field to the axial field is slightly greater than unity, the dispersion and resolving power varies little from that of the straight solenoid, and when this ratio is less than unity the dispersion becomes smaller than that of the straight solenoid spectrometer. The final analysis shows that this combination of magnetic fields fails to give either a two- or one-directional focusing spectrometer of high dispersion and resolving power for non-combined fields.

N. Chako (Flushing, N.Y.)

4739:

Tonks, Lewi. Self-consistent field of single-type electrons in a uniform magnetic field. Phys. Rev. (2) **114** (1959), 637–643.

The properties of electrons rotating in a uniform impressed magnetic field have been of interest lately in connection with work on controlled thermonuclear reactors. The present paper tackles this problem in the case of a somewhat artificial model, where the calculation can be done exactly. The model consists of an infinitely long helix, made up of relativistic electrons in a magnetic field. All electrons have the same mass, axial velocity, and canonical angular momentum. Calculations are carried out for the interesting range of two parameters in the problem. The results are given both in qualitative terms and in quantitative graphs. More realistic approximations are promised to be treated in later papers.

M. J. Moravcsik (Livermore, Calif.)

4740:

Longmire, C. L.; Nagle, D. E.; and Ribe, F. L. Ion confinement by rotation in magnetic mirror geometry. Phys. Rev. (2) **114** (1959), 1187–1191.

The configuration studied in this paper is that of a "magnetic mirror" with a superimposed, generally radial, electric field. It is shown that such a configuration is more suitable to contain plasma than the "unelectrified" mirror machine. The orbits of the single particles in the electrified mirror configuration are discussed, and starting from the exact single-particle Hamiltonian an adiabatic invariant of the drift motion is derived and then used to calculate the guiding surfaces of the ions and the energy balance of their secular motion along the axis of the mirror configuration. Exact numerical calculations of the mirror enhancement are also given.

M. J. Moravcsik (Livermore, Calif.)

4741:

Case, K. M. Plasma oscillations. Ann. Physics **7** (1959), 349–364.

The linearized equation for a plasma (with electrostatic interaction alone) has been solved by means of Laplace transforms by Landau [Acad. Sci. USSR J. Phys. **10** (1946), 25–34; MR **9**, 401]; and by means of normal modes by the reviewer [Physica **21** (1955), 949–963; **23** (1957), 641–650; MR **17**, 690; **19**, 1011]. The velocity distribution of the unperturbed plasma was supposed Maxwellian in the former solution, and spherically symmetrical in the latter one. In the present article a solution in terms of normal modes is presented in which this restriction is removed. As a consequence a number of new normal modes may occur, but it is again possible to prove completeness. It is also shown explicitly that the integral that expresses the final solution as a superposition of normal modes is identical to Landau's Laplace integral, but for the choice of integration path.

N. G. van Kampen (Utrecht)

4742:

★Wait, James R. Electromagnetic radiation from cylindrical structures. Pergamon Press, New York-London-Paris-Los Angeles, 1959. ix+200 pp. $8.00.

This book gives a clear survey of applications of Maxwell's equations to problems of cylindrical symmetry. A number of special cases are worked out, concerning circular and elliptic cylinders, normal wedges and cylindrically-tipped wedges, corner reflectors, knife edges (all having perfectly conducting surfaces), cylinders with dielectric coating, and corrugated cylinders. The sources producing the field are assumed alternately as point or line sources, as a single slot on the surface (with a prescribed distribution of the tangential electric field), an array of slots, or also as a plane wave. Well-known problems, as well as their extensions to more general situations are discussed. Special simple cases are often verified. The general theory for a corner reflector of any angle, e.g., leads to the conventional image theory if this angle becomes a submultiple of 180°.

Cylindrical coordinates z, ρ, φ are used in most chapters. The field components are then expressed in terms of modes of the type:

$$(1) \qquad Z_{m\pi/\psi}\{(k^2-h^2)^{1/2}\rho\}\cdot \exp\{-ihz+im(\pi/\psi)\varphi\},$$

Z being any Bessel or Hankel function. The uniqueness of the solution requires integral values of m, but the continuous spectrum of the other separation variable h involves an integration. The results of chapters 13 and 14 on elliptic cylinders are represented with the aid of Mathieu functions.

The expansions in terms of the modes (1) are derived by starting from a corresponding expansion representing either the source, the distribution of the tangential electric field over the slot, or the solution of a properly chosen application of Green's integral theorem. The convergence of the series in question may be improved by a Watson transformation (known from the diffraction problem of a sphere) converting the original one into a more rapidly converging "residue series" (see chapters 8 and 10). Sometimes, it is preferable to consider the intermediate contour integral leading to the latter series, and to approximate this contour integral with the aid of Sommerfeld's approximation for Hankel functions. This method, introduced by Fock (see chapter 9), can also be developed straightforwardly, starting from a proper approximation of the wave equation; this is discussed in chapter 18. Most final expressions concern radiation patterns describing the field at large distances; they are based on saddlepoint approximations. Corresponding numerical results are represented in a large number of graphs.

The representation is in general very clear, though not on page 63. More references to formulas derived in preceding chapters could have been given. Chapter 2 constitutes a very useful survey of the literature on slotted antennas and related subjects. The book ends in an appendix on rather elementary antenna problems connected with the subject. A factor e_m is missing in formula (133).

H. Bremmer (Eindhoven)

4743:

Collins, W. D. On the solution of some axisymmetric boundary value problems by means of integral equations. I. Some electrostatic and hydrodynamic problems for a spherical cap. Quart. J. Mech. Appl. Math. **12** (1959), 232–241.

The first problem discussed is the determination of the electrostatic potential due to a perfectly conducting spherical cap maintained at a given potential symmetrical about the axis of the cap.

The second deals with the irrotational axisymmetric motion of a perfect fluid past a spherical cap. From this is deduced the effect of the slow steady rotation of a spherical cap in a viscous fluid. *E. T. Copson* (St. Andrews)

4744:

★Wendt, G. Statische Felder und stationäre Ströme. Handbuch der Physik. Herausgegeben von S. Flügge. Bd. 16, Elektrische Felder und Wellen, pp. 1–164. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1958. vii +753 pp. DM 158.00.

Volume 16 of the *Handbuch der Physik*, dealing with "Electric Fields and Waves", is divided into five parts (articles). This first article presents the theoretical aspects of electrostatic and magnetostatic fields due to charges, conductors and dielectric materials. In the first two chapters the author has formulated the general principles and theorems of stationary electric and magnetic fields, vector and scalar potentials, fields produced by multipoles, current distribution in conductors and the application of Green's function to the solution of Poisson's equation. This is followed by a treatment of a variety of two-dimensional potential problems by means of complex variable theory and conformal mapping, including polygonal shaped boundaries (conductors) (Schwartz mapping theorem). In the fourth chapter three-dimensional potential problems are analyzed in great detail for geometries where the potential equation is separable, especially for geometrical arrangements (systems) of conductors which play an essential role in electron optical problems. The mathematical analyses of these problems are rather complete, which should prove very useful to designers of electron optical apparatus. The last chapter contains a treatment of potential fields for geometrical systems (boundary value problems) for which the method of separation of variables is no longer applicable. Here various numerical and graphical methods for finding solutions of the potential equation are described, including relaxation methods, and various experimental methods (electrolytic tank, etc.) are described for field determination for more complex geometrical arrangements of conductors.

In view of the limited space, the author has chosen the topics with great care from the point of view of the theory as well as for practical applications. In the opinion of the reviewer, this article should prove very useful not only as a reference, but likewise as a text for a semester course in this field for engineering and physics students. A well selected bibliography is listed at the end.

N. Chako (Flushing, N.Y.)

4745:

Boerboom, A. J. H.; Tasman, H. A.; and Wachsmuth, H. Shape of the magnetic field between conical pole faces. Z. Naturf. **14a** (1959), 816–818.

"Calculation is made of the shape of the magnetic field between conical pole faces, which may be used as an inhomogeneous deflecting field for a mass spectrometer. The results are expressed as a series expansion in the coordinates around the main path, and in the gap width at the radius of the main path." *Authors' summary*

4746:

★Bremmer, H. Propagation of electromagnetic waves. Handbuch der Physik. Herausgegeben von S. Flügge. Bd. 16, Elektrische Felder und Wellen, pp. 423–639. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1958. vii +753 pp. DM 158.00.

The difficulty of selecting the material on electromagnetic wave propagation for a "Handbuch" article may be well appreciated by taking a glance at the vast literature on this subject which has appeared in the last quarter of a century. In spite of the limited space, the author, who has made important contributions to the field, has done an admirable job in covering most of the important topics falling within the scope of the title of this fourth article in volume 16, which should be very useful not only as a reference for further research, but also for those who plan to pursue their studies in this important field of physics and communication engineering.

After the usual introduction of Maxwell's field equations, energy relations, the Hertzian and Fitzgerald potentials leading to rigorous solutions of the field equations in free space for time dependent and time harmonic fields (monochromatic waves), a discussion of the reduction to a single scalar function satisfying the wave equation, and the representation of the distant field and the radiation properties of the fields in terms of the field distribution on a closed surface, the author discusses the geometrical optical approximation of the wave equation, the equation of transport of the fields along the light rays, following Luneberg's work and Kline's extension for the vectorial case, and gives an illuminating discussion of the application of the method of stationary phase to Kirchhoff's integral (scalar case) for obtaining the geometrical approximation of the fields of caustics or in the vicinity of a focus. The next chapter is devoted to the calculation of the field quantities produced by various kinds of transmitters, such as electric and magnetic dipoles, half-wave dipole, linear arrays, slot and biconical antennas and their properties, aperture antennas and their transmission cross section. After a brief analysis of the geometrical and wave theory of reflectors, especially of a paraboloidal reflector, the author discusses micro-wave lenses and their focusing properties, in particular, the Luneberg lens. The fourth chapter deals mainly with radio wave propagation under the influence of other factors than the curvature of the earth. In the first sections of this chapter the author treats the famous Sommerfeld problem of an electric dipole above the earth, the Zenneck wave, vertical magnetic dipole, and also the total field due to an arbitrary current distribution above the earth. The author then considers the effect of inhomogeneities and roughness of the earth (hills, etc.), reflection from a perfectly conducting terrain and propagation over an inhomogeneous plane earth (land and water) and through a stratified atmosphere, all analyzed from the standpoint

of ray theory and wave theory. An analysis of the problem on the basis of wave theory is given for the approximate solution of the wave equation obtained by the WKB method. The sky wave is also discussed in terms of the geometrical optics approximation to the solution of the wave problem. Furthermore, the mode theory of a stratified atmosphere is discussed in detail. The author then proceeds to discuss the effect of the earth's magnetic field on wave propagation in the atmosphere, turbulence effects and scattering processes for coherent and incoherent scattering and other fluctuation phenomena in the atmosphere which affect radio propagation.

In the last chapter the influence of the curvature of the earth is considered. Here the author gives the rigorous theory of wave propagation around the earth for an homogeneous atmosphere, which is a condensed version of the material given in his book [*Terrestrial radio waves*, Elsevir Publ. Co., Amsterdam-New York, 1949; MR **11**, 295]. This is followed by discussion of the effect of an inhomogeneous curved atmosphere, and the modal theory of a curved stratified ionosphere, etc. Finally, the author gives a survey of various phenomena associated with radio propagation through the curved atmosphere, such as duct propagation, diffraction and twilight zones in the troposphere, long and medium wave propagation in the ionosphere, short wave propagation via the ionosphere etc.

In conclusion, the reviewer wishes to recommend this article to persons working in this field, especially chapter IV, where they will find a wealth of material that is scattered in a number of periodicals and has now been put in a coherent and readable form.

N. Chako (Flushing, N.Y.)

4747:

★Кузнецов, П. И.; и Стратонович, Р. Л. Распространение электромагнитных волн в многопроводных системах. Сборник статей. [Kuznecov, P. I.; and Stratonovič, R. L. Propagation of electromagnetic waves in multiconductor systems. A collection of articles.] Vyčislitel'nyi Centr, Akad. Nauk SSSR, Moscow, 1958. vii+85 pp. Distributed without charge.

"The six articles in this collection are concerned with several basic problems relating to the propagation of electromagnetic waves in multiconductor systems. Article 1 presents a quasi-exact method of solution for the case where the parameters characterizing the system satisfy the inequalities

$$|\chi_e^2/\chi_i^2| \ll 1, \qquad a_n \leqq |\chi_e^{-1}|,$$
$$|(\mu_i'\chi_e^2)/(\mu_e'\chi_i^2)| \ll \mathrm{Min}(1, 1/a_n),$$

where the subscripts e and i refer to the exterior and interior media, χ is the critical value of the propagation constant, μ' is the complex permeability, and a_n is the radius of nth conductor. The method involves a successive approximation technique in which the skin depth is taken as a small parameter. The solution for the general case is applied in Article 2 to the important special case of symmetrical and non-symmetrical two-conductor systems in free space. Article 3 is concerned with the transient state of multiconductor systems, the starting point being a generalized form of telegraphist's equation. The method developed here is applied in Article 4 to the analysis of a finite line terminated in a finite load. Article 5 considers the representation of contour integrals of the form

$$\int_\gamma F(u, \sqrt{(Au^2+Bu+C)}) \exp \frac{z}{2}(au+b+\sqrt{(Au^2+Bu+C)})du$$

in terms of cylindrical functions of one and two arguments. Here F is a rational function in its arguments, a, b, A, B, C are constants, z is a parameter and γ is a finite closed contour containing all singularities of the integrand. In Article 6 two cylindrical functions defined by the series

$$\Gamma_n(w, z) = \sum_{m=0}^{\infty} \left(\frac{w}{z}\right)^{n+2m} I_{n+2m}(z),$$

$$\theta_n(w, z) = \sum_{m=0}^{\infty} \left(\frac{z}{w}\right) I_{n+2m}(z),$$

where $I_n(z)$ is a modified Bessel function of the first kind, are studied in detail and tabulated." (From the authors' summary) *L. A. Zadeh* (Berkeley, Calif.)

4748:

Veklenko, V. A. Green's function for the resonance radiation diffusion equation. Soviet Physics. JETP **36** (9) 1959, 138–142 (204–211 Ž. Eksper. Teoret. Fiz.).

"An analytic expression for the Green's function of the resonance radiation diffusion equation has been derived for the case of a homogeneous infinite space. The properties of the Green's function have been investigated for dispersion and Doppler spectral lines. An analytic expression has been derived for the mean time required for a photon to move as a result of diffusion over a distance greater than some prescribed value. In conclusion, the Green's function has been determined for the stationary equation and its asymptotic expression is given in explicit form for a dispersion spectral line." *Author's summary*

4749:

Bobrinev, V.; and Braginskii, V. The radiation of a point charge moving uniformly along the axis of a circular hole in an infinite ideally conducting plane. Soviet Physics. Dokl. **123** (3) (1958), 1179–1181 (634–636 Dokl. Akad. Nauk SSSR).

4750:

★Hartshorn, L.; and Saxton, J. A. The dispersion and absorption of electromagnetic waves. Handbuch der Physik. Herausgegeben von S. Flügge. Bd. 16, Elektrische Felder und Wellen, pp. 640–725. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1958. vii+753 pp. DM 158.00.

This article is devoted to the experimental methods for measuring the dielectric constant and absorption coefficients of dielectric substances in the long and short wave length region. Several methods and kinds of apparatus for measuring these constants for gases, liquids and solids are described, namely the alternating current procedure using alternating current bridges (low frequencies) and also direct currents, where the relaxation time is long. In the high frequency region, several methods are described employing cavity resonators and simple wave guide structures. These experiments are based on the observation of the change of resonance frequency of the resonator

resulting from the introduction of the substances in the resonators. When wave guide structures are employed, the standard procedure for determining the constants of liquid and gases is based on the theory of standing waves. From the form of the standing wave pattern and the location of the maxima and minima of the field one can determine the value of the reflection coefficient and the phase of the waves by properly matching the impedances of the system. This method is especially suited for liquids, but can be used for determining the dielectric properties of gases and solids as well, at frequencies in the centimeter range. For shorter wave lengths, interferometers, similar to those used in optical measurements are employed. The most useful of these are the micro-wave version of the Michelson and the Fabry-Perrot interferometer. The final chapter is devoted to a discussion of the dielectric properties of mixtures and the results obtained from experiments, especially organic liquids. A list of books dealing with the theoretical and also with the experimental determination of the dielectric properties of substances is included. *N. Chako* (Flushing, N.Y.)

4751:

de Hoop, A. T. On the plane-wave extinction cross-section of an obstacle. Appl. Sci. Res. B **7** (1959), 463–469.

The extinction cross section of an obstacle is defined in this paper as the sum of its absorption and scattering cross sections, which characterize the time average powers absorbed and scattered relative to the time average intensity of power flow in the incident time harmonic, plane wave field. An important linear relationship that exists between the extinction (or total) cross section and the amplitude of the scattered wave in the direction of incidence has been appreciated and utilized for some years. This paper provides another derivation of the explicit formulas in the case of (vector) electromagnetic and (scalar) sound fields, with the help of far field representations of the scattered waves.

H. Levine (Stanford, Calif.)

4752:

Miyamoto, Kenyō. On a comparison between wave optics and geometrical optics by using Fourier analysis. III. Image evaluation by spot diagram. J. Opt. Soc. Amer. **49** (1959), 35–40.

With the help of some of the results derived in two previous papers [same J. **48** (1958), 57–63, 567–575; MR **19**, 1009; **20** #2974] image evaluation based on spot diagrams is considered. It is shown how the geometrical optics response function may be directly calculated from the coordinates of the spots. Evaluation of the "Strehl definition" on the basis of geometrical optics is also discussed. Finally the effect of the degree of coherence in the object plane on the intensity distribution in the image plane is briefly considered. *E. Wolf* (Rochester, N.Y.)

4753:

Senior, T. B. A. The scattering of electromagnetic waves by a corrugated sheet. Canad. J. Phys. **37** (1959), 787–797.

The reflection of an incident plane wave field propagating in the negative y direction (normal incidence) with its magnetic vector entirely in the z direction by a corrugated sheet is investigated. The corrugations are parallel to the z-axis and the contour of the sheet has the equation $y=a\cos(Kx)$. The scattered field is produced by the current distribution excited on the surface of the sheet. According to physical optics a reasonable approximation can be found by assuming that the distribution at every point of the surface is the same as if the reflection were taking place at an infinite tangent plane in this point. The scattered field (magnetic component also in the z direction only) is expressed as a double integral. An asymptotic evaluation by the saddle point method (for large distances of the point of observation from the sheet) reduces this to a representation as an angular spectrum of plane waves, which is shown to be valid for all distances of the point of observation from the sheet.

F. Oberhettinger (Corvallis, Ore.)

4754:

Krivoglaz, M. A.; and Rybak, S. A. Contribution to the theory of scattering of light near points of second-order phase transitions. Soviet Physics. JETP **6** (1958), 107–117.

"The scattering of light by ferroelectric and ferromagnetic crystals in the vicinity of second-order transition points is considered." *From the authors' summary*

4755:

Spruch, Larry. Bounds on scattering phase shifts. Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. **CX-30** (1957), i+25 pp.

4756:

★Borgnis, F. E.; and Papas, C. H. Electromagnetic waveguides and resonators. Handbuch der Physik. Herausgegeben von S. Flügge. Bd. 16, Elektrische Felder und Wellen, pp. 285–422. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1958. vii+753 pp. DM 158.00.

The present article, the third in volume 16 of the *Handbuch der Physik*, is concerned with the theoretical aspects of electromagnetic waves in cavities and wave guide structures. Its primary purpose is to give an adequate account of those parts of the field which are associated with simple geometrical structures manageable to mathematical analysis, nowadays familiar to advanced students and to persons with a good engineering or physics background. In this respect, this article could form a good foundation for a semester course for graduate classes in physics or electrical engineering.

Beginning with Maxwell's equations set up in symmetric form by the introduction of a fictitious magnetic current density and the boundary conditions, the authors give a classification of the type of waves which are supported within a perfectly conducting tube filled with a homogeneous dielectric (source free) from the nature of the solutions of Maxwell's equations obtained with the Hertzian potential. This is followed by a discussion of the orthogonality properties of the fields (transverse and longitudinal components) and the analogy with transmission line theory. The third chapter contains a detailed analysis of various kinds of wave guides filled with an isotropic and lossless medium whose boundaries are parts of curves of a coordinate system in which the wave

equation is separable. Among such structures are included the elliptic and parabolic (cross section) wave guides and triangular shapes, the sectorial and other forms of wave guides amenable to exact analysis; and also a brief discussion of a wave guide of arbitrary cross section is given. In the following chapter are included analytic solutions of the sectorial horn, the so-called quasi-pyramidal horn, conical horn and geometrical structures which by conformal transformation of the wave equation become separable, curved wave guides and in particular the circular dielectric wave guide, which should find important applications to the new branch of fiber optics. The following two chapters are concerned with waves guided along surfaces of conductors, such as wires, corrugated structures with circular or rectangular cross sections or in the form of a helix. Of particular importance is the detailed discussion of the properties of the field in wave guide junctions, analyzed from the transmission line analogy. The concluding chapter is devoted to cavity resonators. Here one finds a clear exposition of the effect of a wave guide connected with a cavity on the resonance frequency, impedance and fields of the resonator. Finally, the authors give a list of books which partially or in full treat the subject matter of this article.

N. Chako (Flushing, N.Y.)

4757:

Wait, James R.; and Mientka, Walter E. Calculated patterns of slotted elliptic-cylinder antennae. Appl. Sci. Res. B **7** (1959), 449–462.

The azimuthal radiation pattern of a narrow slot of finite length cut in the axial direction of a perfectly conducting elliptic-cylindrical surface is expressed as an infinite series involving Mathieu functions. This pattern is numerically evaluated for a number of combinations of parameters. The results are shown in five tables and numerous diagrams, indicating that for "reasonably large elliptic cylinders the patterns are dependent mainly on the surface curvature in the neighborhood of the slot".

C. J. Bouwkamp (Eindhoven)

4758:

Wait, James R. Earth currents near a monopole antenna with symmetrical top loading. J. Res. Nat. Bur. Standards **62** (1959), 247–255.

The general expressions are found for the field produced by a vertical groundbased monopole with umbrella top loading with an assumed current distribution in the antenna. The umbrella loading is represented by a cone coaxial with, and connected to, the top of the monopole. The H-field produced by the monopole is checked in the near field at the base of the antenna, where it is shown to correspond to Ampère's law, and in the far-zone, where the first term of an expansion gives the familiar dipole radiation field. Graphs are presented for the earth current distribution below a disk loaded monopole with an assumed parabolic dependence upon radius of the current in the disk. The earth current is found to be modified only insignificantly from that of a monopole with the same base current. The corresponding expression for the far zone shows only a small modification from the field of the vertical monopole at disk radii small compared to a wavelength.

An expansion of Green's function e^{-ikR}/R, which is obtained in the paper because, as stated, "it does not seem to be available", can be found, e.g., in Morse and Feshbach, *Methods of theoretical physics*, Vol. 2 [McGraw-Hill, New York, 1953; MR **15**, 583; p. 1466].

H. A. Haus (Vienna)

4759:

Weibel, Erich S. Stable orbits of charged particles in an oscillating electromagnetic field. Phys. Rev. (2) **114** (1959), 18–21.

Orbits of a charged particle in the electromagnetic field in a circular wave guide driven in the TE_{01}-mode at cutoff are investigated. Conditions under which the particle will be stably bound to the axis are derived.

F. Oberhettinger (Corvallis, Ore.)

4760:

Haines, M. G. The inverse skin effect. Proc. Phys. Soc. **74** (1959), 576–584.

"The axial current density in an infinite cylindrical conductor is calculated as a function of radius and time. In the problem studied the total axial current is specified as a function of time. The case of a rigid conductor is studied first, and the analysis is extended to the case of a radially expanding or contracting cylinder such as occurs in a linear pinched plasma. The radial velocity is assumed proportional to the radius, while the outside radius of the conductor is a general but specified function of time.

It is shown that the variation of total current with time can give rise to distributions of current density which bear no resemblance to the normal skin effect. In fact, an inverse skin effect or even reversal of current may occur. This latter effect causes a reversed pinch force and perhaps eventual break-up near the surface of a plasma. Some special total current profiles are used to illustrate these features." *Author's summary*

4761:

van Gelder, A. P.; de Graaf, A. M.; and Kronig, R. New calculations on the Faraday effect in wave guides. Appl. Sci. Res. B **7** (1959), 441–448.

The authors compute the Faraday rotation that may be produced by a circular waveguide coated adjacent to the wall with a thin uniform layer of gyromagnetic medium whose gyroaxis is parallel to the axis of the guide. They place in evidence certain advantages of locating the gyromagnetic medium along the wall rather than along the axis of the guide in the form of a concentric rod.

C. H. Papas (Pasadena, Calif.)

4762:

★**King, Ronold W. P. Quasi-stationary and non-stationary currents in electric circuits.** Handbuch der Physik. Herausgegeben von S. Flügge. Bd. 16, Elektrische Felder und Wellen, pp. 165–284. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1958. vii+753 pp. DM 158.00.

This article, the second of volume 16 of the *Handbuch der Physik*, contains a condensed but clear exposition of quasi-stationary and non-stationary electric fields. Starting with a discussion of the concepts of surface and volume charge and current densities, polarization and magnetization, the author has formulated the general laws and properties of electromagnetic fields, boundary relations between the field quantities across surfaces separating two media with different electric and magnetic properties

and energy and power equations in terms of density and current distributions. By expressing the field quantities in terms of scalar and vector potentials and using Green's theorem, integral expressions for the potentials and energy and power of the field are derived in terms of Green's function, for an arbitrary space distribution of charge and current. The author then proceeds to analyze the external and internal (skin effect) fields in cylindrical conductors, circular condensors, coupled circuits and two terminal networks. In the following chapter the theory of transmission lines is presented in some detail, especially the two-wire terminal lines and two-terminal pair networks. The properties of such systems (impedance, resonance and voltage and current distributions and transfer of power and discontinuity effects) are analyzed. The last two chapters are primarily devoted to radiating systems (antenna theory). Beside the different types of antennas, the author analyzes in great detail the non-resonating loop antenna, centrally loaded cylindrical receiving antenna and other loop antennas which are used as probes for measuring fields. Several diagrams representing the properties of some antenna systems and results obtained from experiments are included in the text. Finally, a selected bibliography of general works treating partly or in a comprehensive manner the topics discussed in this article is listed. *N. Chako* (Flushing, N.Y.)

4763:

Nagumo, Zin-ich. On pseudo-harmonic oscillations of third order. Proc. Fac. Engrg. Keio Univ. **10** (1957), 27–43.

Using well-known perturbation methods (Coddington and Levinson's generalization of Poincaré's method), the author studies the oscillations of weakly nonlinear systems of the third order. The results are applied to self-sustained and to synchronized Colpitts oscillators and to Parametron circuits, and there are consequences which the author says are of practical importance.

J. P. LaSalle (Baltimore, Md.)

4764:

Possenti, Renzo. Sul principio di successione in alcuni problemi elettrodinamici. Atti Accad. Naz. Lincei. Mem. Cl. Sci. Fis. Mat. Nat. Sez. I (8) **5** (1959), 47–58.

Every science presents not only problems unsolved in consequence of their practical difficulty, but also problems the solution of which is impossible because it would violate some logical or physical law. The author gives some examples of the latter category in electrodynamics, that he calls idealistic, suggesting that utopistic would be a better name. The logical principle violated in each of them is the principle of succession, that is, that the effect cannot precede the cause. In all these examples this violation of the principle of succession is accompanied by one of the relation between the real and the imaginary part of the function that expresses the relation between effect and cause. *H. Bremekamp* (Delft)

4765:

Chirlian, Paul M. Restrictions imposed upon the unit step response of linear phase shift networks. Quart. Appl. Math. **17** (1959), 225–230.

The transfer function of an electrical network without phase distortion has the form $T(\omega)e^{-jk\omega}$, $T(\omega)\geqq 0$, $k\geqq 0$. The author considers the response of such a network to unit step input and obtains lower bounds for the 10%–90% rise time r_t and upper bounds for the overshoot o_s. If there exists a cut-off frequency ω_c, such bounds are obtained in closed and easily evaluated form in terms of ω_c and $\max T(\omega)/T(0)$. It is shown that the bounds obtained in this case are best possible, and that they may be used as approximations to the bounds when $\lim_{\omega\to\infty} T(\omega)=0$. Other bounds when there is no cut-off frequency involve one or the other of $\int_0^\infty T(\omega)d\omega$, $\int_0^\infty [T(\omega)]^2 d\omega$, and are valid if this integral exists. It is shown that a result of A. H. Zemanian on a related problem [Proc. I. R. E. **43** (1955), 322–326; MR **16**, 777; Theorem I] can be modified to show that if $T(\omega)$ is monotone decreasing and $\lim_{\omega\to\infty} T(\omega)=0$, then $o_s\leqq .0895$.

It is assumed in bounding o_s that $T(\omega)$ is such that the network is physically realizable. The bounds for r_t are derived with the assumption that there exists no 10% response point after the time $t=k$. The author points out that this condition is usually fulfilled in useful circuits. It would have been helpful to note that it is certainly fulfilled if $T(\omega)/\omega$ is non-increasing.

R. Cohn (New Brunswick, N.J.)

4766:

Duffin, R. J. Distributed and lumped networks. J. Math. Mech. **8** (1959), 793–826.

The problem of finding the conductances of a distributed network is equivalent to solving Dirichlet's problem. Starting from the distributed network the author introduces two lumped networks (upper and lower networks) whose conductances give bounds for the original network. These have the usual advantages that lumped circuits have over distributed circuits and in addition are well suited to numerical treatment. The case of planar networks is covered and relations between the conjugate network and the upper and lower networks are discussed.

J. Blackman (Syracuse, N.Y.)

CLASSICAL THERMODYNAMICS, HEAT TRANSFER

See also 4689, 4695, 4700.

4767:

★Pippard, A. B. Elements of classical thermodynamics for advanced students of physics. Cambridge University Press, New York, 1957. vii + 165 pp. Paperbound: $2.75.

This textbook is intended to fulfil the need of physics and mathematics students for a short account of the fundamental ideas of thermodynamics which can supplement a brief course of lectures. It is available in a well bound and printed paper-backed edition which is comparatively cheap.

The plan of the book is as follows. The laws of thermodynamics are introduced in the first four chapters, and are followed by two chapters containing a miscellany of ideas and applications. Chapter 7 is entitled "The Thermodynamic Inequalities" and is mainly a discussion of thermodynamic stability. Chapter 8 is devoted to phase equilibrium, with special reference to helium and superconductors, and chapter 9 is an account of higher-order transitions. Fourteen fairly difficult exercises are included.

The author has indeed achieved very well the stated aims of his preface. Nevertheless, as is inevitable in a short account, the book is open to criticism in some respects. After the first few chapters the development of the subject can hardly be said to be orderly. Instead of making for the fundamental thermodynamic equations for a system and emphasizing that all the thermodynamic properties can be derived and related by means of them, various applications of thermodynamics, either of general importance or of special interest in contemporary physics, are discussed. Among these applications, that to magnetizable bodies is especially clear and valuable.

A second criticism is that certain sections, and in particular the last chapter, are too detailed for such a small book and upset its balance as a textbook. These are no doubt included deliberately in order to get the student within hailing distance of contemporary interests in physics in this field. However, the reviewer doubts the wisdom of attempting to take the student so rapidly from the fundamentals to the more esoteric applications. His scientific education suffers because he is not familiar with the thermodynamic description and analysis of a host of common phenomena. He knows the phase diagram of helium but not that of water. Every substance he handles in the laboratory is impure to some extent, and he ought to know in what way its properties are affected.

A third criticism, related to the last, is the absence of any mention of chemical potentials. This seems to the reviewer to be a mistake for two reasons. First of all because the chemical potential really belongs to the fundamentals of thermodynamics (and has nothing especially "chemical" about it in origin) and secondly because of its occurrence in the grand ensemble of statistical mechanics. It is surely desirable nowadays to emphasize those features of classical thermodynamics which are especially important in statistical thermodynamics.

It is only to be expected that the author of a new textbook on a well known subject has undertaken his task partly because he has his own views about presentation. The author has quite a lot of new ideas about presenting thermodynamics, and most of them seem excellent. Among them is a discussion of adiabatic changes in the manner suggested by Carathéodory before introducing the second law, but without becoming committed to Carathéodory's principle as a statement of this law. Another is the refreshingly sensible discussion of the law of increase of entropy and fluctuations.

The book can be warmly recommended to teachers and students of physics. *W. Byers Brown* (Manchester)

4768:

Concer, D. B. Heat flow towards a moving cavity. Quart. J. Mech. Appl. Math. **12** (1959), 222–231.

A function V satisfies the conditions $V_{xx}+V_{yy}+cV_x=0$ when $r>r_0$, $V=V_0$ when $r=r_0$, and $V\to V_R$ when $r\to\infty$, where $r^2=x^2+y^2$ and $c=U/D$. Here U, D, V_0 and V_R are prescribed constants. The author solves that boundary value problem for $V(x, y)$ in the form of a series involving Bessel and trigonometric functions. The function V represents steady temperatures in the unbounded medium exterior to the cylinder $r=r_0$ as that cylinder moves with velocity U parallel to the x axis. The problem is associated with heat flow toward a stope in a mining operation. The author studies the flow of heat into the cylindrical wall. He also solves the corresponding problem for the moving elliptical cylinder. *R. V. Churchill* (Ann Arbor, Mich.)

4769:

Rektorys, Karel. Determination of the temperature in a solid gravity dam when inner heat-sources are considered. Rozpravy Československé Akad. Věd. **66** (1956), no. 14, 74 pp. (5 unbound inserts) (Czech. Russian and English summaries)

4770:

Stier, Fr. Die stationäre, radiale Wärmeströmung in einem Hohlzylinder für beliebige Grenzbedingungen. Arch. Elektrotech. **44** (1959), 271–274.

4771:

Surinov, Iu. A. On certain questions in the theory of radiation transfer and radiant heat exchange in an absorbing medium. Soviet Physics. Dokl. **123** (3) (1958), 1136–1140 (813–816 Dokl. Akad. Nauk SSSR).

4772:

Tirskiĭ, G. A.; and Trenogin, V. A. Determination of the temperature field of a cooled blade of a gas turbine. Izv. Akad. Nauk SSSR Otd. Tehn. Nauk Energet. Avtomat. **1959**, no. 2, 45–48. (Russian)

4773:

Klimov, D. M. Damping of proper oscillations of a gyroscope in Cardan's suspension with dry friction. Soviet Physics. Dokl. **123** (3) (1958), 1141–1143 (410–412 Dokl. Akad. Nauk SSSR).

QUANTUM MECHANICS

See also 4264, 4663, 4664.

4774:

Coish, H. R. Elementary particles in a finite world geometry. Phys. Rev. (2) **114** (1959), 383–388.

This is an attempt to derive the quantum numbers of elementary particles from the Lorentz group by means of altering the mathematical structure of space. The continuously varying coordinates x_μ are replaced with numbers from the field of integers modulo some prime p. These numbers can only locally be ordered, but that is sufficient for agreement with present observational data if p is chosen very large (of order $10^{10^{31}}$). The Lorentz group is replaced with the finite group of linear transformations leaving $x_0^2-x_1^2-x_2^2-x_3^2$ invariant. A natural extension of this group obtains by including the transformations that change the sign of this form. A spinor group, homomorphic $(p+1)$ to one, can be found, and from it the irreducible representations are built up in the usual way. Because of the $(p+1)$-valuedness of the representation a discrete gauge transformation of the spinors is possible, which gives rise to a quantized charge. The two simplest ways of realizing a single charge give rise to

particles of spin 0 and spin $\frac{1}{2}$, which the author tentatively identifies with the π^+ meson and the Σ^+ particle. Space inversion automatically changes the sign of the charge. It is mentioned that only V and A interactions are covariant with respect to the extended group of linear transformations. *N. G. van Kampen* (Utrecht)

4775:

Stevens, K. W. H.; and Josephson, B., Jr. The coupling of a spin system to a cavity mode. Proc. Phys. Soc. **74** (1959), 561–575.

"The problem of a non-interacting spin system coupled to a damped resonant cavity is formulated wave-mechanically. An exact solution of the Schrödinger equation is found for this system, and the expectation value M_z of the z component of total spin angular momentum S_z is determined in series form from an expansion of the solution. The expectation value is obtained exactly and in closed form only for very small values of the total spin quantum number S. Approximate solutions in closed form are obtained for the initial conditions $M \simeq S$ and $M \simeq -S$. For the former condition, $\langle t|S_z|t\rangle$ exhibits exponential decay for small values of t. The approximations break down for larger values of t, but damped oscillatory behaviour is suggested. The results are discussed in terms of the total number of spins, and a value of at least 10^{18} spins seems necessary for an appreciable exponential change in $\langle t|S_z|t\rangle$ to occur, under typical experimental conditions. Reasons are given why a simple, closed form for $\langle t|S_z|t\rangle$ seems unlikely for the general case of arbitrary S, and some mathematical difficulties are considered in more detail." *Authors' summary*

4776:

Tietz, T. The calculation of the differential elastic cross section for complex atoms for the self consistent field. Acta Phys. Acad. Sci. Hungar. **10** (1959), 251–252.

4777:

Klein, Abraham; and Zemach, Charles. Analytic properties of the amplitude for the scattering of a particle by a central potential. Ann. Physics **7** (1959), 440–455.

Previous results of these authors on the convergence of the Born series [Zemach and Klein, Nuovo Cimento (10) **10** (1958), 1078–1087; MR **21** #580] are used to give a new proof of dispersion relations for potential scattering at fixed momentum transfer. It is also shown that the partial wave expansion can be used to define the scattering amplitude in the unphysical region in precisely the same domain of momentum transfer for which the dispersion relations are valid. These results apply to both Schrödinger and Klein-Gordon equations.

J. C. Polkinghorne (Cambridge, England)

4778:

Buchdahl, H. A. On the trace of the energy-momentum tensor of fields associated with particles of zero rest mass. J. Austral. Math. Soc. **1** (1959/61), part 1, 99–105.

The author proves the theorem that the trace of the energy-momentum tensor of all fields whose Lagrangian is a conformal invariant vanishes, either identically or as a result of the field equations. He then shows that this theorem applies in particular to all fields of integral and half-odd spin corresponding to particles with zero rest mass, with the spin-zero field the only exception.

P. G. Bergmann (New York, N.Y.)

4779:

Ruijgrok, Th. W. Exactly renormalizable model in quantum field theory. III. Renormalization in the case of two V-particles. Physica **25** (1959), 357–364.

[For parts I, II, see Physica **24** (1958), 185–204, 205–213; MR **20** #693, #694.] The author continues his earlier work and investigates the properties of a model of a quantized field theory which was introduced by him and van Hove a few years ago. In part II, the author wrote down a Schrödinger equation for this model where only renormalized quantities appeared. This Schrödinger equation is further investigated and simplified in the paper reviewed here. In particular, the author investigates states with more than one heavy particle. It is conjectured that the method developed here may also be useful in more realistic cases. *G. Källén* (Lund)

4780:

Van Hove, Léon. Strong and weak interactions in a simple field-theoretical model. Physica **25** (1959), 365–370.

The author investigates the field theoretical model earlier introduced by him and Ruijgrok. This model is a generalization of the well-known Lee model and contains two kinds of heavy particles, V_1 and V_2. These heavy particles interact with a relativistic boson (θ-particle). The model contains certain selection rules which are constructed in such a way that the two transitions $V_1 \rightleftarrows V_2+\theta$, $V_2 \rightleftarrows V_1+\theta$ are allowed. The interaction is described by two unrenormalized coupling constants $g_1{}^0$ and $g_2{}^0$. It has been known for some time [cf. Ruijgrok and Van Hove, Physica **22** (1956), 880–886; MR **18**, 626] that the two renormalized coupling constants g_1 and g_2 can be defined in such a way that they become equal in the limit of no cut off. In this case, the model contains no difficulties with negative probabilities and anomalous states (ghosts). The author now makes the very interesting remark that if one introduces a very large but finite cut off energy, the model contains two effective coupling constants that are nearly but not quite equal. This can be interpreted in such a way that the model describes two kinds of interactions which are different in character. There is a strong interaction and a weak interaction and the weak interaction is caused by the difference of the two coupling constants g_1 and g_2. The model has a complete symmetry between V_1 and V_2 if the weak interaction is neglected. This symmetry is destroyed by the weak interaction. The author emphasizes that these properties of the model appear as a consequence of the interaction and are not introduced in the original Hamiltonian. *G. Källén* (Lund)

4781:

Katayama, Yasuhisa; Taketani, Mituo; and Leal Ferreira, Jorge. On the origin of the electron mass. Progr. Theoret. Phys. **21** (1959), 818–826.

The authors search for possible means of explaining (i.e., calculating) the mass of the electron in terms of the concepts of current quantized field theories. Their first

procedure is to express a structure theory for the electron by means of a non-local formulation of quantum electrodynamics. This requires an energy cut-off in the neighborhood of 10^4 Mev. An attempt is made also to relate the electron mass to the masses of the mesons via weak interactions between electrons and mesons. It is concluded that such an explanation of the electron mass can be decided only by more complete knowledge of high energy interactions in the region of 10^6 Mev.

E. L. Hill (Minneapolis, Minn.)

4782:

Leal Ferreira, J.; and Zimerman, A. H. Application of Dirac's operational method to a system of two fermions. An. Acad. Brasil. Ci. **30** (1958), 281–285.

This paper is concerned with obtaining the radial equations for a two particle Dirac equation in which the free particle Hamiltonians are additive and the total angular momentum is a constant of the motion. This problem has previously been treated by expanding the sixteen-component wave function in terms of spherical harmonics [see, for example, R. Arnowitt and S. Gasiorowicz, Phys. Rev. (2) **94** (1954), 1057–1062]. For the case of the Breit potential the authors propose an alternate operational procedure for obtaining the radial equations. Here the Dirac matrices of the two particles that enter into the wave equation generate an algebra with only 64 basis elements and hence can be represented by 8×8 matrices. A new constant of the motion is then constructed by making use of this basis. This new operator is the two particle generalization of the Dirac relativistic one body operator $\beta(\sigma \cdot L+1)$. The radial equations can then be obtained in a simple fashion as in the one body case.

R. Arnowitt (Syracuse, N.Y.)

4783:

Sekine, K. On the renormalization of a parity non-conserving interaction. Nuovo Cimento (10) **11** (1959), 87–101. (Italian summary)

A Lorentz invariant quantum field theory describing (for instance) interacting nucleons, hyperons, and mesons and containing parity non-conserving terms is studied from the viewpoint of renormalization theory. It is shown that the formally exact procedure of Källen [Helv. Phys. Acta. **25** (1952), 417–434; MR **14**, 435] can be applied to obtain equations which define the various renormalization constants. When these equations are solved in lowest order perturbation theory, they turn out to be consistent only if the parity non-conserving parts of the theory vanish. It is not clear whether this result persists in higher order.

A. Klein (Philadelphia, Pa.)

4784:

Novožilov, Yu. V. Clothed particles in the quantum field theory. Vestnik Leningrad. Univ. **13** (1958), no. 22, 93–106. (Russian. English summary)

The aim of the paper is to give a formulation of quantum field theory in terms of physical or dressed particles. The procedure followed is to introduce for each particle physically present a complete set of creation and destruction operators which are used to describe the cloud surrounding the particle. Thus for a pion-nucleon scattering problem one introduces a double set of operators for pions, nucleons and antinucleons, the operators of the first set commuting with those of the second. The correspondence between the states in this "double" representation and the states in the usual representation is given. The interaction between fields can be expressed in terms of an effective interaction hamiltonian relating to physical particles.

L. Van Hove (Utrecht)

4785:

Olsson, Per O. M. Transition matrix elements between states in a Coulomb field. II. Ark. Fys. **15** (1959), 289–301.

A continuation of previous studies [P. O. M. Olsson, same Ark. **15** (1959), 159–167; MR **21** #1845b] of integrals containing products of confluent hypergeometric functions. As is well known [A. Erdélyi, Math. Z. **42** (1936), 125–143] these integrals may be evaluated in terms of generalized hypergeometric series. From the point of view of physical applications, the principal difficulty remaining is the useful quantitative evaluation of these generalized hypergeometric functions. The main point of the present paper is to derive a number of transformations of these functions which lead to series expansions that converge rapidly when the parameters and variables lie in the range of physical interest.

T. Erber (Chicago, Ill.)

4786:

Olsson, Per O. M.; and Hultberg, Sölve. A theoretical study of the internal conversion with special application to magnetic dipole transitions. Ark. Fys. **15** (1959), 361–386.

In a series of earlier papers [see #4785 above] one of the authors has shown how the transition matrix elements between states in a Coulomb field may be evaluated in terms of certain generalized hypergeometric functions. By taking advantage of the specific simplifications which occur for the case of internal conversion, the computation of the internal conversion coefficients may be carried out explicitly. Near the conversion threshold, the hypergeometric series may be evaluated numerically by convenient approximations; there is good agreement with the corresponding results of Rose, Goertzel, Spinrad, Han and Strong [Phys. Rev. (2) **83** (1951), 79–87]. The relativistic wave functions (bound and continuum) are then modified to first order to take into account finite nuclear size corrections; the modified results are also in excellent agreement with the work of Sliv [*Tables of internal conversion coefficients of gamma rays*, Izdat. Akad. Nauk SSSR, Moscow-Leningrad, 1958].

T. Erber (Chicago, Ill.)

4787:

Tomishima, Yasuo. Thomas-Fermi-Dirac theory with correlation correction. Progr. Theoret. Phys. **22** (1959), 1–11.

"Using the formula of the correlation energy due to Pines, we modify the Thomas-Fermi-Dirac model with the Fermi-Amaldi correction for free atom or ion to include correlations between electrons."

From the author's summary

4788:

Olsen, Haakon; and Maximon, L. C. Photon and electron polarization in high-energy bremsstrahlung and pair production with screening. Phys. Rev. (2) **114** (1959), 887–904.

The authors' extensive work on bremsstrahlung and pair production is well known. The present paper is another one in the series. The authors calculate the polarization of electrons and photons, taking into account Coulomb and screening effects exactly. The formulae apply to arbitrary electron polarization and for plane or circularly polarized photons. Graphs are given, and the qualitative features of the results are also summarized. Finally, the electron spin-photon polarization correlation is discussed and the depolarization of the electrons is also calculated. *M. J. Moravcsik* (Livermore, Calif.)

4789:

Laurikainen, K. V. Deuteron wave functions and phenomenological two-nucleon potentials. Ann. Acad. Sci. Fenn. Ser. A. VI. no. 27 (1959), 20 pp.

This paper presents a full discussion of some calculations of deuteron wave functions for various nucleon interaction potentials. Both central and non-central potentials are considered, and also potentials containing a hard-core term, and a spin-orbit interaction term.

A variation method is used throughout, and some discussion is given of the most appropriate trial functions to use in each case. The results obtained are applied in a discussion of the Gammel and Thaler potential, and of the effects of spin-orbit and hard-core interaction on the deuteron ground state. The method used appears to deal with any potential made up of terms of the form $(e^{-ar}r^{-n})$, though difficulties have been found in choosing trial functions in complicated cases. *D. F. Mayers* (Oxford)

4790:

Holøien, Erling. The use of the associated Laguerre functions in the method of configurational interaction. Norske Vid. Selsk. Forh., Trondheim **31** (1958), no. 2, 8 pp.

A rather succinct account of the use of associated Laguerre functions in the calculation of the energy levels of atoms including helium-like ions and lithium. Included in the paper are definitions of the Laguerre functions and tables of results and comparison of these with the Hyllerass' work over which it seems to represent a rather significant improvement. However the details of the method are completely omitted.

H. Feshbach (Cambridge, Mass.)

4791:

Kisdi, D. The statistical model nuclei with angular momentum. Acta Phys. Acad. Sci. Hungar. **10** (1959), 29–38. (Russian summary)

A statistical model of nuclei has been developed by Gombás [e.g. Fortschr. Physik **5** (1957), 159–174]. His work applies to nuclei with zero angular momentum in the stable equilibrium state. The present work extends this theory to non-zero angular momenta, following the work on the corresponding atomic problem by Sessler and Foley [Phys. Rev. (2) **96** (1954), 366–368]. The results are used, in conjunction with experimental work by Druin, Polikanov, and Flerov [Ž. Eksper. Teor. Fiz. **32** (1957), 1298–1304] on nuclear fission, to predict the minimum mass number of a nucleus which will spontaneously disintegrate even with zero angular momentum. Good agreement is found with other determinations.

M. J. Moravcsik (Livermore, Calif.)

4792:

Rayski, J. A six-dimensional interpretation of nuclear forces. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 251–254. (Russian summary, unbound insert)

4793:

Mahmoud, Hormoz. Angular distribution of polarized electrons emitted from oriented nuclei. Ann. Physics **7** (1959), 429–439.

A procedure first used by Rose and Osborn is applied to the calculation of the angular distribution of polarized electrons emitted from oriented nuclei. The final result is an exact expression in which the only unknown quantities are the reduced matrix elements. The latter can be evaluated only after a definite nuclear model is adopted.

Author's summary

4794:

★Ядерные реакции при малых и средних энергиях. [Nuclear reactions with small or moderate energies.] Proceedings of the All-Union conference, Nov., 1957. Izdat. Akad. Nauk SSSR, Moscow, 1958. 614 pp. (1 plate, 1 insert) 32.60 rubles.

4795:

Hamada, Tetsuo. Kinematics of the neutron-deuteron scattering. Progr. Theoret. Phys. **21** (1959), 769–778.

This paper is a study of the kinematics of n-d scattering, including spin-orbit coupling. In this case l and s are not good quantum numbers, and hence the scattering matrix is of the multichannel type. The matrix, nevertheless, can be parametrized in terms of real and independent parameters. The differential cross section and the polarization are written down in terms of these, and the low energy behavior is studied. This latter can be written down in terms of two scattering lengths. The formalism is not valid without modification where inelastic processes are important. *M. J. Moravcsik* (Livermore, Calif.)

4796:

Garibian, G. M. Radiation from a charged particle traversing a layered medium. Soviet Physics. JETP **35** (8) (1959), 1003–1006 (1435–1439 Ž. Eksper. Teoret. Fiz.).

One of the significant technical problems associated with the study of elementary particles is the construction of more and more sensitive detectors, particularly in the highly relativistic region. The author considers the very interesting possibility of enhancing the transition radiation emitted by a charged particle which passes through a slab of material by introducing a series of parallel slabs separated by vacuum. It is found that under certain circumstances the radiation is in fact proportional to the number of slabs. *L. Spruch* (New York, N.Y.)

4797:

Bollini, C. G. Irreducibility constraints and field equations for the elementary particles. Nuovo Cimento (10) **11** (1959), 342–350. (Italian summary)

A tensor $A_{\mu_1\dots\mu_s}$ can be the wave function of an elementary particle of spin s if it belongs to an irreducible representation of the homogeneous Lorentz group. Sufficient conditions for this are the following. (a) $A_{\mu_1\dots\mu_s}$ is completely symmetric in its indices, (b) $A_{\mu\mu\mu_2\dots\mu_s}=0$, and (c) $\partial_{\mu_1}A_{\mu_1\dots\mu_s}=0$. The author succeeds in constructing a basis system of tensors $p^{(\rho)}_{\mu_1\dots\mu_s}$ such that the most general $A_{\mu_1\dots\mu_s}$ has the form

$$A_{\mu_1\dots\mu_s}=\sum_\rho A^{(\rho)}p^{(\rho)}_{\mu_1\dots\mu_s}.$$

The "components" $A^{(\rho)}$ are taken as the independent components of the wave function, and action principles for free and interacting particles are stated. The commutation relations in the quantized version are also derived in a simple and straightforward manner.

C. A. Hurst (Adelaide)

4798:

van Dam, H.; Hilgevoord, J.; and Wouthuysen, S. A. Some recent developments in the theory of elementary particles. Nederl. Tijdschr. Natuurk. **25** (1959), 245–255. (Dutch)

4799:

Ganguly, S. Production of mesons in nucleon-nucleon collisions and its dependence on collision cross-sections. J. Tech. Bengal Engrg. Coll. **3** (1958), 81–91.

The well-known method of Mellin transforms is used to solve the integrodifferential equations describing the growth of a nucleon-meson cascade in homogeneous nuclear matter and the atmosphere. The decay of mesons is taken into account and the results compared with those of previous workers. *H. Messel* (Sydney)

4800:

Ritchie, R. H. Interaction of charged particles with a degenerate Fermi-Dirac electron gas. Phys. Rev. (2) **114** (1959), 644–654.

A quantum-mechanical theory of the interaction of a charged particle with a degenerate electron plasma is presented. The reaction of the plasma to the field of the incident particle is described in terms of a dynamical dielectric constant for the plasma. The interaction of the particle and the plasma is formulated in terms of Feynman diagrams, with the dielectric constant operator playing the role of an interaction operator. It is shown that this theory yields results equivalent to those of present perturbation theories in the limiting cases of small and large energy transfer between particle and plasma. It has potential usefulness in many other applications. *E. L. Hill* (Minneapolis, Minn.)

4801:

Gell-Mann, M. The interpretation of the new particles as displaced charge multiplets. Nuovo Cimento (10) **4** (1956), supplemento, 848–866.

4802:

May, Robert M. Pair correlations in the quasichemical equilibrium approach to superconductivity. Progr. Theoret. Phys. **22** (1959), 12–14.

It is found that before a quantitative theory of superconductivity can be derived from the quasichemical equilibrium approach, a self-consistent method of evaluating the "quenched" correlation matrix for electron pairs is needed. *Author's summary*

4803:

Bardeen, J.; Rickayzen, G.; and Tewordt, L. Theory of the thermal conductivity of superconductors. Phys. Rev. (2) **113** (1959), 982–994.

A theory of the thermal conductivity of superconductors is presented, based on the theory of superconductivity due to Bardeen, Cooper, and Schrieffer. The excited states of the system are treated as quasi-particles, allowing a Boltzmann equation to be set up. The electronic contribution to the thermal conductivity when the dominant scatterers are impurities has been calculated exactly. The result is very close to that of the Heisenberg-Koppe theory which is in fair agreement with experiment. The variational principle of Wilson has been used to find the electronic conductivity when the dominant scatterers are lattice waves. It is concluded that the theory fails to predict the sharp drop in the ratio k_{es}/k_{en} as the temperature is lowered below T_c, a feature which is characteristic of the experimental results. The effect of the electrons on the lattice conductivity has also been calculated. The theoretical values may be too large.

L. N. Cooper (Providence, R.I.)

4804:

Pines D.; and Schrieffer, J. R. Gauge invariance in the theory of superconductivity. Nuovo Cimento (10) **10** (1958), 496–504. (Italian summary)

It is shown that when due account is taken of the Coulomb interactions between electrons, a theory of superconductivity which gives a satisfactory account of the Meissner effect in the London gauge can be extended to give a satisfactory account in any other gauge. Calculations based on the Bardeen-Pines collective treatment of the electron-phonon system show that in the long wave length region ($k\lesssim k_c\simeq r_s^{-1}$, where r_s is the interelectronic spacing) only the plasmons respond to a longitudinal vector potential. Their response is such as to maintain gauge invariance. The individual electrons are surrounded with a screening cloud of virtual plasmons in such a way that they carry no longitudinal current to order (k^2/k_c^2), and are hence unaffected by a longitudinal vector potential to this order. The introduction of the screening cloud is shown to correspond closely to the backflow introduced by Feynman and Cohen to guarantee longitudinal current conservation for the excitations in liquid helium. The calculations are carried out to order m/M and k^4/k_c^4, and involve the use of the random phase approximation for terms which couple plasmons and electrons. For the wave lengths of importance in the Meissner effect ($k\ll k_c$), these approximations are well-justified. An explicit prescription is given for extending the Bardeen, Cooper and Schrieffer derivation of the Meissner effect to an arbitrary gauge, and their results are shown to be essentially unaltered.

L. N. Cooper (Providence, R.I.)

RELATIVITY

See also 4732, 4733, 4774.

4805:

Pignedoli, Antonio. Über die Bewegung von relativistischen Elektronen in elektrischen und magnetischen Feldern. Z. Angew. Math. Mech. **38** (1958), 343–348.

Scopo della nota è la discussione del problema del moto di un elettrone in un campo di forza di Lorentz nello ambito della teoria della relatività speciale. L'A. discute anche il moto in un campo che si obtiene soprapponendo un campo elettrico e un campo magnetico, quando ciascuno di questi soddisfa a particolari condizioni strutturali. *G. Lampariello* (Rome)

4806:

Rancoita, G. M. Forze in un corpo polarizzato e magnetizzato. Nuovo Cimento (10) **11** (1959), supplemento 183–224.

In questa notevole memoria l'A. propone un nuovo tensore energetico nell'ambito della elettrodinamica relativistica della materia polarizzata e magnetizzata immersa in un campo elettromagnetico.—È ben noto che la questione presenta difficoltà che sono ben lungi dall'esser superate [cfr. A. Sommerfeld, *Electrodynamics*, Academic Press, New York, 1952; MR **14**, 433]. La proposta dell'A. viene posta a confronto con quelle di Minkowski e Dällenbach, di Abraham, di Kneissler. Lo scopo della teoria è la determinazione delle forze elettromagnetiche che si esercitano nei corpi polarizzati e magnetizzati.

G. Lampariello (Rome)

4807:

Israel, W. Discontinuities in spherically symmetric gravitational fields and shells of radiation. Proc. Roy. Soc. London. Ser. A **248** (1958), 404–414.

Into Riemannian space-time the author introduces a class of coordinates (L'-coordinates) related to Lichnerowicz's admissible coordinates [A. Lichnerowicz, *Théories relativistes de la gravitation et de l'électromagnétisme*, Masson, Paris, 1955; MR **17**, 199; Chapter I] by transformations which are C^1, piecewise C^2. Using L' coordinates he derives the O'Brien-Synge junction conditions [Stephen O'Brien and John L. Synge, Comm. Dublin Inst. Adv. Studies. Ser. A. no. 9 (1952); MR **14**, 913] from Lichnerowicz's continuity conditions. This settles the question, whether the latter include the former, in the affirmative, as was to be expected. He then discusses spherically symmetric space-times, and in particular expanding spherical shells of (non-gravitational) radiation, using L'-coordinates. His results contradict the surprising conclusion of Raychaudhuri [Z. Physik **135** (1953), 225–231; MR **15**, 168] that the inner boundary of such a shell overtakes the outer boundary.

F. A. E. Pirani (London)

4808:

Treder, H. Stosswellen des Gravitationsfeldes. Ann. Physik (7) **2** (1958), 225–235.

Stellmacher [Math. Ann. **115** (1938), 741–783] discussed discontinuities in the gravitational field from the point of view of the theory of characteristics of Einstein's vacuum field equations. Here the author derives Stellmacher's results in a more concise and elegant manner by working directly with the Riemann tensor instead of with the second derivatives of the metric tensor, and extends his method to the case where an electromagnetic field is present. The results agree with those of Bel [C. R. Acad. Sci. Paris **245** (1957), 2482–2485; MR **19**, 1237], Trautman [Bull. Acad. Polon. Sci. A. III. **5** (1957), 273–277; C. R. Acad. Sci. Paris **246** (1958), 1500–1502; MR **19**, 509, 1237] and the reviewer [Phys. Rev. (2) **105** (1957), 1089–1099; MR **20** #3020] wherever they overlap.

F. A. E. Pirani (London)

4809:

Sachs, R.; and Bergmann, P. G. Structure of particles in linearized gravitational theory. Phys. Rev. (2) **112** (1958), 674–680.

The metric = the Lorentz metric + power series in parameter λ. Working to first order in λ, the authors define "total energy" and "total angular momentum" which are "gauge invariant", that is, invariant to transformations $x'^\rho = x^\rho + \lambda\xi^\rho(x)$, ξ^ρ arbitrary. Solutions of the linearized Einstein equations in de Donder coordinates are classified according to the total energy and total angular momentum of their sources (world tubes containing matter or field singularities). In this approximation the sources do not interact, and the motion of each is independent of its quadrupole and higher moments; however the latter may be used to refine the classification, in terms of the (gauge-invariant) Riemann tensor.

The difficulty of developing similar techniques for the full theory is discussed. *F. A. E. Pirani* (London)

4810:

Finzi, Bruno. Movimento gravitazionale. Rend. Sem. Mat. Fis. Milano **28** (1959), 61–77. (English summary)

In questo lavoro che è stato oggetto di una Conferenza dell'A. vengono esposte le linee fondamentali della teoria di Einstein del moto gravitazionale iniziata nel 1938 con l'importante Memoria di Einstein, Infeld ed Hoffmann [Ann. of Math. (2) **39** (1938), 65–100]. Lo scopo della teoria è di mostrare come le equazioni del moto gravitazionale si possono dedurre dalle equazioni di Einstein del campo gravitazionale considerando i corpi in movimento come singolarità puntiformi del campo gravitazionale stesso. La teoria ha ricevuto nuovi importanti contributi da Einstein e il suo collaboratore L. Infeld nelle Memorie [ibid. **41** (1940), 455–464; Canad. J. Math. **1** (1949), 209–241; MR **1**, 283; **11**, 59]. Le osservazioni fatte dall'A. costituiscono un'ottima introduzione allo studio delle Memorie originali di Einstein la cui interpretazione presenta notevoli difficoltà.

G. Lampariello (Rome)

4811:

Møller, C. The energy-momentum complex in the general theory of relativity. Mat.-Fys. Medd. Danske Vid. Selsk. **31**, no. 14, 39 pp. (1959).

The author had suggested previously [Ann. Physics **4** (1958), 347–371; MR **20** #732] that the canonical stress-energy complex of Einstein and Tolman be replaced by a new expression, four components of which transform with respect to arbitrary curvilinear transformations of the three "spatial" coordinates among themselves. In this

paper it is shown that this new expression is the quasi-canonical stress-energy complex associated with the variational principle whose integrand is $(-g)^{1/2}R$. The discussion is fairly broad, so that the paper may also be read by non-specialists.

P. G. Bergmann (New York, N.Y.)

4812:

Araki, Huzihiro. On weak time-symmetric gravitational waves. Ann. Physics **7** (1959), 456–465.

L'A. montre qu'il existe une solution et une seule des "conditions initiales" du cas extérieure à dérivées temporelles nulles qui soit conforme à une métrique donnée voisine de la métrique de Minkowski et à comportement asymptotique minkowskien. La démonstration repose sur un théorème global d'existence et d'unicité pour l'équation:

$$\Delta_g\Phi + R_g\Phi = 0.$$

[Δ_g et R_g Laplacien et courbure de la métrique donnée; cf. Lichnerowicz, J. Math. Pures Appl. (9) **23** (1944), 37–63; MR **7**, 266; Fourès-Bruhat, J. Rational Mech. Anal. **4** (1956), 951–966; MR **18**, 976.]

L'A. compare avec la métrique de Schwarzschild le comportement asymptotique de la métrique obtenue et montre qu'à celle-ci on peut considérer comme associeé une énergie positive. *Y. Fourès-Bruhat* (Reims)

4813:

Moffat, John. On the integrability conditions in the problem of motion in general relativity. J. Math. Mech. **8** (1959), 771–785.

A review, from various sources, of the title problem, plus isolated original remarks which make no contribution to its solution. *F. A. E. Pirani* (London)

4814:

Gottlieb, I. L'étude du champ mésonique dans la lumière de la théorie de la relativité générale. An. Şti. Univ. "Al. I. Cuza" Iaşi. Secţ. I. (N.S.) **4** (1958), 89–94. (Russian. Romanian and French summaries)

4815:

Geissler, D.; Papapetrou, A.; und Treder, H. Die Gravitationsstrahlung eines zeitweilig nichtstationären Systems. Ann. Physik (7) **2** (1958), 344–350.

4816:

Finlay-Freundlich, E. The empirical foundation of the general theory of relativity. Scientia (6) **53** (1959), 181–187.

4817:

Geissler, D.; und Treder, H. Über ebene Wellen in der allgemeinen Relativitätstheorie. Tensor (N.S.) **8** (1958), 165–168.

Takeno [Tensor (N.S.) **7** (1957), 97–102; MR **21** #2515] exhibited a solution of Einstein-Maxwell field equations for a source-free electromagnetic field

$$((g_{ij})) = \begin{pmatrix} -A & & & \\ & -A & & \\ & & -B & \\ & & & B \end{pmatrix},$$

$$((F_{ij})) = \begin{pmatrix} 0 & & -\sigma & \sigma \\ & & \rho & -\rho \\ \sigma & -\rho & & \\ -\sigma & \rho & & 0 \end{pmatrix},$$

where A, B, ρ, σ are functions of $Z \overset{\text{def}}{=} z-t$. The field equations, in this case, reduce to one:

$$(1) \qquad A''-(A')^2/2A-A'B'/B = -8\pi(\rho^2+\sigma^2),$$

where $(\ )' \overset{\text{def}}{=} d(\ )/dZ$. In the paper under consideration the authors prove that (a) a suitable choice of coordinates brings B to the form $F(A)$ where $F \neq 0$ is arbitrary, and, in particular, the equation (1) reduces for $B=F(A)=1$ to (2) $A^{1/2}(A^{1/2})'' = -4\pi(\rho^2+\sigma^2)$; (b) the solution A of (2) is not physically meaningful.

V. Hlavatý (Bloomington, Ind.)

4818:

Venini, Carlo. Sul vettore distribuzione elettrica nella teoria relativistica unitaria Einsteiniana. Ist. Lombardo Accad. Sci. Lett. Rend. A **92** (1957/58), 228–239.

L'auteur étudie les équations de la dernière théorie d'Einstein (dans le vide) en première et en seconde approximation; le choix de coordonnées isothermes au premier ordre, qui le restent au second, est une conséquence des équations du champ.

Il montre, par l'étude du vecteur distribution électrique, que la séparation au premier ordre des champs gravitationnel et électromagnétique n'est plus valable au second ordre; il y existe donc une interaction, et un champ neutre au premier ordre peut avoir une charge et un courant non nuls au second ordre. Dans le cas particulier des champs stationnaires, un champ pur, magnétique ou électrique au premier ordre peut le rester au second. Les équations relatives à ces champs sont explicitement formées.

L'auteur applique ces résultats au champ magnétique et électrique terrestre et détermine ainsi les ordres de grandeur des champs électrique et magnétique terrestre du second ordre. *J. Renaudie* (Montpellier)

4819:

Eisenhart, Luther P. Spaces for which the Ricci scalar R is equal to zero. II. Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 226–229.

L'A. détermine explicitement des solutions de l'équation $R=0$ (R courbure riemannienne scalaire) dans un espace à trois dimensions en coordonnées orthogonales, en vue de la détermination de solutions symétriques dans le temps des équations d'Einstein.

Y. Fourès-Bruhat (Reims)

4820:

Jankiewicz, C. Über den Zusammenhang der vom Feld abhängigen Eigenmasse mit dem konform-Minkowskischen Raume. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 765–769.

Szamosi [Acta Phys. Acad. Sci. Hungar. **6** (1956), 207–215; MR **18**, 445] and Infeld [Bull. Acad. Polon. Sci. Cl.

III. 5 (1957), 491–495; MR **19**, 814] have derived equations of motion for a particle whose proper mass depends on the (scalar) field in which it moves. In the present paper it is shown that these equations are the equations of a geodesic in a conformally Minkowskian space-time C_4. The author assumes a variational principle which leads also to the vanishing of the scalar curvature of C_4, and thence deduces Poisson's equation for the scalar field. {He appears to be unaware of the result of Levinson and Zeisler [*Law of gravitation in relativity*, University of Chicago Press, 1931; pp. 89 and 100], rediscovered by the reviewer [Proc. Cambridge Philos. Soc. **51** (1955), 535–537; MR **16**, 1165] that geodesic motion in a C_4 of zero scalar curvature implies a planetary perihelion motion of 1/6 the observed amount, in the wrong direction.}

F. A. E. Pirani (London)

4821:

Weissmann, A. Über den anisotropen Charakter der verallgemeinerten relativistischen Räume. Nuovo Cimento (10) **9** (1958), 1016–1026.

The space of the title is a "generalized Riemann space" in the sense used by Eisenhart [Proc. Nat. Acad. Sci. U.S.A. **37** (1951), 311–315; **38** (1952), 505–508; **39** (1953), 546–551; MR **13**, 279; **14**, 317, 1123], that is, a space with non-symmetric fundamental tensor, which in the present paper is assumed to have the form* $g^{\mu\nu} = g^{\mu\nu} - \varepsilon\phi^{\mu\nu}$, where $g^{\mu\nu}$ is symmetric, and ε is a small parameter. The properties of such a space are discussed, using analogies with, and some of the language of, the theory of small deformations of a continuous medium [compare J. Rumer, Nachr. Ges. Wiss. Göttingen Kl. II **16** (1931), 148–156].

F. A. E. Pirani (London)

ASTRONOMY

See also 4579.

4822:

★**Finlay-Freundlich, E. Celestial mechanics.** Pergamon Press, New York-London-Paris-Los Angeles, 1958. viii + 150 pp. $7.50.

In this short book on Celestial Mechanics the introductory and first chapters give a succinct, well-written review of the equations of motion and their classical solution for the two-body problem as well as for the n-body problem with special reference to the restricted problem of three bodies. A brief but useful picture is given of the singularities of the three-body problem and Sundman's treatment thereof. In particular, in the simple case of two mass-points moving collinearly, it is shown how the motion may be regularized, leading to a solution which may be extended beyond the collision point.

Chapter II is devoted to the Hamilton-Jacobi method. The particular advantages of canonical elements and transformations are explained and stressed in applications to perturbation theory. This is a welcome discussion in simple terms of material that very often is presented as mere facts without enlightening motivation and prospective utility. The two-body problem is worked out in canonical variables as an example to illustrate the method and to point out the relations between the dynamical problem and the canonical invariants. This is especially useful to indicate how the perturbations are introduced and treated in the more general problem.

Chapter III is titled "The Theory of Perturbations". Here vast areas of perturbation theory are skimmed over in a few sentences while other important parts, such as Hansen's Method, are ignored altogether. However, the author concentrates on the core of perturbation theory, looking at the forests and neglecting the trees.

After deriving the canonical equations in the "Jacobi-Radau" variables to indicate one method for simplifying the equations, the author goes into general considerations. Here he discusses the existence of solutions in power series in the disturbing mass and points out the limitation that the mass places upon the interval of time within which the expansions remain valid. The chapter concludes with a discussion of secular and periodic perturbations touching upon the form of the terms in the Fourier expansions. Finally, Delaunay's method is very briefly described.

Chapter IV contains cursory information on a few of the well-known inequalities in the motion of the moon. It is shown why the moon's motion is so complicated and how the high degree of accuracy is attainable. The chapter concludes with a short account of certain periodic solutions in the three-body problem as well as mention of the relativistic advance of Mercury's perihelion.

In Chapter V the two-body problem for deformable bodies is discussed in much detail because of the application to close binary systems which can give a great deal of information concerning the internal constitution of stars. The basic assumption is made that each component has time to adjust to the equilibrium configuration defined at each instant at time. Unlike previous theories, the eccentricity is introduced. This turns out to produce second-order effects which change the motion of periastron by a factor of 2.5, similar in character to the effect in the lunar theory of the neglect of the sun's motion about the earth-moon system. After developing that part of the disturbing potential which gives rise to rotational and tidal deformations, the solution is carried out on the basis of certain simplifying assumptions that affect to a minor extent only the accuracy of the results. Finally the equations are used to evaluate $k^{(2)} = (\varepsilon - \frac{1}{2}\phi)/\phi$, where $\phi = \omega^2 a^3/GM$ for the binary system Υ Cygni, showing excellent agreement with Chandrasekhar's theory.

The final chapter VI deals with the relativistic advance of the line of apsides. This is done in the usual manner by establishing the geodesic of the four dimensional line element ds, which is simply assumed from relativity theory. An appended note briefly describes the orbital motion of stars in globular clusters.

The artificial satellite problem is not treated in this volume on "Celestial Mechanics".

M. S. Davis (New Haven, Conn.)

4823:

Littlewood, J. E. Corrections to the paper "On the problem of n bodies". Kungl. Fysiogr. Sällsk. i Lund Förh. **29** (1959), 97–98.

Original is Comm. Sém. Math. Univ. Lund [Medd. Lunds Univ. Mat. Sem.], Tome Supplémentaire (1952), 143–151 [MR **14**, 910].

4824:

Phillips, John G. The application of digital computers to the reduction and analysis of high-dispersion molecular spectra. Astrophys. J. **130** (1959), 308–323.

"Programs have been developed for the IBM 701 digital computer for the calculation of vacuum wave numbers from line measurements and for the identification of the branches into which the lines of individual bands may be divided. Tests on representative spectra show that great savings of time and labor may be achieved, especially if the programs are used in conjunction with digitizing equipment on the measuring machines to punch the measurements directly onto IBM cards." (Author's abstract) *C. B. Haselgrove* (Manchester)

4825:

Krook, Max. Structure of stellar atmospheres. III. Astrophys. J. **130** (1959), 286–295.

[For parts I and II, see same J. **122** (1955), 488–497; **129** (1959), 724–733; MR **17**, 543; **21** #2527.] The structure equations for model stellar atmospheres have generally to be solved by means of an iteration procedure. A double-moment method is presented for the solution of the equation of transfer, within the framework of such an iteration scheme. *Author's summary*

4826:

Kaminisi, Keisuke. On the adiabatic change of completely ionized gases in the stars. Kumamoto J. Sci. Ser. A. **2** (1956), 358–362.

4827:

★Bracewell, Ronald N. (Editor) Paris symposium on radio astronomy. International Astronomical Union symposium No. 9 and International Scientific Radio Union symposium No. 1, held from 30 July to 6 August, 1958. Stanford University Press, Stanford, Calif., 1959. xii+612 pp. $15.00.

The editor of this volume deserves congratulations for this speedy publication of the proceedings of the symposium. The main contributions were delivered by invited speakers under the heads: I Moon and Planets; II The Sun; III Galactic and Extragalactic Radio Sources; IV The Large-scale Structure of Galaxies; V Discrete Sources and the Universe; VI Mechanisms of Solar and Cosmic Emission. A record of discussion following the contributions ensures that there are no great gaps in this presentation as an overall survey of the fast-growing subject of radio astronomy.

The theorist will note with interest the rise of the synchrotron process of emission as a likely source of non-thermal radiation. The account of recent Russian work on the older mechanism of plasma oscillations is also valuable. *K. C. Westfold* (Sydney)

4828:

Arsac, J. Essai de détermination des très faibles diamètres apparents en radioastronomie. Opt. Acta **6** (1959), 77–98.

"On account of their low resolving power, radio telescopes give images in which there is sometimes only one available datum. (Curvature of the interferometric curve in the region of the origin, or spread of the central maximum in the diffraction pattern.) From this datum are deduced the limits to the apparent diameter of a star, no assumptions being made about the shape of the star. For this purpose use is made of the theory of distributions which is described in general terms." (Author's summary) *K. C. Westfold* (Sydney)

4829:

Arsac, J. Correction approchée de l'effet de lobe en radioastronomie. Opt. Acta **6** (1959), 103–116.

The image of an object seen by an instrument used in optics or radio astronomy is incapable of complete restoration, since certain Fourier components are filtered out by the instrument. The best one can do is to modify the actual image to the form which would be obtained from an instrument having some desired response within the same pass band as the actual instrument. The author shows how to approximate to this form by means of a finite Fourier series. The method is compared with previous techniques, and numerical application is made to problems associated with different instruments used in radio astronomy. *K. C. Westfold* (Sydney)

GEOPHYSICS

See also 4679.

4830:

Aubert, Eugene J.; Lund, Iver A.; and Thomasell, Albert, Jr. Some objective six-hour predictions prepared by statistical methods. J. Meteorol. **16** (1959), 436–446.

Linear regression methods were used to predict pressure, temperature, precipitation, cloudiness and visibility. The factor analysis method was used to select from a large number of possible predictors those which had a significant effect upon the variance of the predictands. As an alternate method, a screening procedure was used to select from the same set of starting predictors those which accounted for most of the predictand variance. The screening procedure consists of representing the normalized predictand as a linear combination of orthonormalized predictors. These predictors are chosen from the set of normalized predictors in decreasing order of their correlation with the predictand down to a cut-off point.

No significant difference in the quality of the forecasts obtained by using the two methods was observed. The screening method has the advantage of simplicity. Stratification of the data on the basis of "wet" and "dry" days had no important effect on the forecasts. However, addition of cloudiness and precipitation to the list of predictors was significant.

J. F. Blackburn (Yorktown Heights, N.Y.)

4831:

Nemčinov, S. V. Short-term prediction of the field of atmospheric pressure for a three-layer model of the atmosphere. Izv. Akad. Nauk SSSR. Ser. Geofiz. **1959**, 432–444. (Russian)

En supposant le mouvement d'air dans l'atmosphère libre comme adiabatique et quasi géostrophique, l'auteur

établit les deux équations fondamentales de l'hydrodynamique du géopotentiel ϕ avec les conditions aux limites: pour $\zeta=0$; $\tau=0$ et pour $\zeta=1$; $\tau=0$ (ζ la pression réduite et τ la vitesse verticale).

On résoud le système numériquement en décomposant l'atmosphère en $2n-1$ couches équidistantes et en remplaçant les dérivées par des différences finies. (Dans le cas étudié $n=2$.) En outre on linéarise le système. Pour résoudre on utilise la méthode de Fourier. L'auteur étudie après la stabilité de la solution. Comme application l'auteur étudie en détail le cas des trois couches de 250 mb, 500 et 750 mb.

Le tracé d'une seule carte a nécessité le choix de 135 points. Pour chaque prévision il a fallu effectuer environ 94.770 opérations, d'où la nécessité d'employer des machines électroniques à grande vitesse.

M. Kiveliovitch (Paris)

4832:

Takeuchi, H. General solutions of equations of some geophysical importance. Bull. Seismol. Soc. America **49** (1959), 273–283.

4833:

Satô, Yasuo. Numerical integration of the equation of motion for surface waves in a medium with arbitrary variation of material constants. Bull. Seismol. Soc. America **49** (1959), 57–77.

The author presents a numerical method for determining the dispersion of both Love and Rayleigh waves in heterogeneous media. For Love waves he holds fixed the wave number and varies the frequency in small steps until the displacement at infinite depth tends to zero. The Rayleigh wave case requires a two parameter search; he does not describe this search procedure. Numerical examples are presented, but no mathematical proofs are given. *J. Douglas, Jr.* (Houston, Tex.)

4834:

Arnold, Kurt. Zur Bestimmung der Geoidundulationen aus Freiluftanomalien. Deutsche Akad. Wiss. Berlin. Veröff. Geodät. Inst. Potsdam no. 12 (1959), 3–50.

Jeffreys [Gerlands Beitr. Geophys. **31** (1931), 378–386] has shown that the use of free-air gravity anomalies in Stokes' formula yields geoidal heights correct to the first order, without any special assumptions concerning the distribution and density of masses within the earth's crust. Arnold now examines the higher-order corrections by carrying out the same development farther. These higher-order terms allow for: (1) the use of a sphere rather than an ellipsoid in the analysis of the potential field; (2) the irregular mass distributions in the earth's crust as a function of the geoidal heights to the first order. The terms (1) are not explicitly developed, being of the order of 1/300 of the geoidal height. Rather complicated expressions are derived for the corrective terms (2), the total magnitude of which usually amounts to a few centimeters, and hardly ever more than a decimeter. Levallois [C. R. Acad. Sci. Paris **244** (1957), 2479–2482; MR **19**, 370] recently derived a similar solution much more simply, but in which the higher order terms are not explicitly exhibited as in Arnold's work.

B. Chovitz (Washington, D.C.)

4835:

Arnold, Kurt. Schwerewerte in grossen Höhen über der Erdoberfläche. Deutsche Akad. Wiss. Berlin. Veröff. Geodät. Inst. Potsdam no. 12 (1959), 51–69.

The author first derives, from Green's second identity and geometrical considerations, an integral formula for the gravity anomaly at a point P exterior to the earth. The formula is a function of the height of P above the earth, the geocentric latitude of P, and the irregularity of the topography. The last-named variable is expressed in terms of the angle of inclination, ν, of the topography to the horizon at any point on the earth's surface, over which the integration is performed. A second formula is then obtained, under the assumption $\nu=0$, which transforms the integral formula into an expression in spherical harmonics. *B. Chovitz* (Washington, D.C.)

OPERATIONS RESEARCH, ECONOMETRICS, GAMES

See also 4401, 4571.

4836:

★Chakravarty, S. The logic of investment planning. Contributions to Economic Analysis, XVIII. North-Holland Publishing Co., Amsterdam, 1959. xii+170 pp. $4.00.

The title of this book might suggest that the subject matter be that of evaluating and comparing investment projects by means of discount formulae or the like. This is, however, not the case. The author has discussed and summarized certain general principles of using dynamic planning models on a national scale. The dynamic element is represented by the accumulation of capital from one period to the next. The discussion centers around models previously developed by Leontief, Tinbergen, Mahalanobis and others. The author has two principal suggestions of his own to improve on such models. One suggestion is to take into account the important lag between the time of investment expenditures and the time of actual availability of the capital items produced. The other suggestion is that the models should allow for "structural breaks", that is, for new industries coming into, or going out of, the picture during the planning period considered.

T. Haavelmo (Oslo)

4837:

Rosett, Richard N. A statistical model of friction in economics. Econometrica **27** (1959), 263–267.

"The statistical model presented in this paper is a generalization of the model of a limited dependent variable described by Tobin [Econometrica **26** (1958), 24–36; MR **19**, 819]. The Tobin model deals with relationships in which it is known that the conditional cumulative distribution function of the dependent variable has a mass point at some lower or upper limiting value of the dependent variable. The present model includes cases in which the mass point is anywhere in the cumulative distribution function . . . This, therefore, is a model of relationships in which the dependent variable is subject to friction." The maximum likelihood method for estimating relationships of this type is presented. "As in the case of the Tobin model, the resulting equations are nonlinear, and the

suggested method of solution is exactly the same as that described by Tobin."

M. J. Beckmann (Providence, R.I.)

4838:

★**Guilbaud, G. Th. Approximation entropique et plans économiques de rangement.** Le calcul des probabilités et ses applications. Paris, 15–20 Juillet 1958, pp. 143–147. Colloques Internationaux du Centre National de la Recherche Scientifique, LXXXVII. Centre National de la Recherche Scientifique, Paris, 1959. 196 pp.

Some probabilistic assignment problems are discussed. In certain cases these lead to the ubiquitous functional $H=\sum p_i \log p_i$ (with a suitable base) as an expression for the minimum cost. *V. E. Beneš* (Murray Hill, N.J.)

4839:

Gazis, Denos C.; Herman, Robert; and Potts, Renfrey B. Car-following theory of steady-state traffic flow. Operations Res. 7 (1959), 499–505.

It is assumed here that the acceleration of any nth car in a sequence at time $t+\Delta$ is proportional to the difference in velocities of the nth and $(n-1)$th cars at time t and inversely proportional to the spacing between the cars at time t. Integration of this relation gives a logarithmic dependence of the velocity upon spacing which in turn yields a steady-state flow vs. density relation that agrees very well with experimental data. This is an improvement of earlier linear theories in which the acceleration was independent of spacing.

G. Newell (Providence, R.I.)

4840:

Morris, William T. Some analysis of purchasing policy. Management Sci. 5 (1959), 443–452.

A given amount of a commodity is to be procured on or before day N. The daily prices x_k $(k=1, \cdots, N)$ are non-negative and independently distributed, with known density functions f_k; the actual value of x_k becomes known on day k. The following strategy minimizes the expected total expense: on day k, procure the whole amount or nothing, according as $x_k \leqq X_k$ or $x_k > X_k$; where $X_N=\infty$, $X_{N-1}=Ex_{N-1}$,

$$X_{k-1} = X_k \int_{X_k}^{\infty} f_k(x_k)dx_k + \int_0^{X_k} x_k f_k(x_k)dx_k \qquad (k = 1, \cdots, N-1).$$

An equivalent recursive formula is given in the paper. The strategy of "dollar cost averaging" is also discussed, and rejected. *J. Marschak* (New Haven, Conn.)

4841:

Musk, F. I. A Monte Carlo simulation of a production planning problem. Comput. J. 2 (1959), 90–94.

4842:

Amato, Vittorio. L'esponenziale di Poisson e la distribuzione del numero dei morti per giorno. Statistica. Bologna 19 (1959), 20–59.

4843:

Charnes, A.; and Cooper, W. W. Nonlinear network flows and convex programming over incidence matrices. Naval Res. Logist. Quart. 5 (1958), 231–240.

The authors handle the case of nonlinear convex functions by means of the usual linear approximations. Combining this technique with a number of other linear programming methods, they are able to treat successfully a number of minimization problems which are not easily treated by any other means.

R. Bellman (Santa Monica, Calif.)

4844:

denBroeder, G. G., Jr.; Ellison, R. E.; and Emerling, L. On optimum target assignments. Operations Res. 7 (1959), 322–326.

"This note is concerned with two target assignment models. An optimum assignment is one which maximizes the expected value of targets destroyed. The first model, which admits an explicit solution, associates values only with the number of targets destroyed. An algorithm which enjoys a computational nicety is established when the values of the individual targets are assumed known. This latter model is a special case of Flood's target-assignment model." (Authors' summary)

M. M. Flood (Ann Arbor, Mich.)

4845:

Abrham, Jaromír. Über eine Verallgemeinerung des Transportproblems. Časopis Pěst. Mat. 84 (1959), 183–187. (Czech. Russian and German summaries)

The author solves the generalized transportation problem in linear programming on the assumption that the supply exceeds the demand. The problem is to find the minimum of the linear form $\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$ with respect to the set of all non-negative matrices (x_{ij}) of the type m, n, for whose elements $\sum_{i=1}^{m} x_{ij}=b_j$ $(j=1, \cdots, n)$, $\sum_{j=1}^{n} x_{ij} \leqq a_i$ $(i=1, \cdots, m)$ holds, $a_1, \cdots, a_m$, $b_1, \cdots, b_n$ being such given positive numbers that $\sum_{i=1}^{m} a_i > \sum_{j=1}^{n} b_j$. It is shown that the problem may be reduced to the problem in which $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$.

D. Mazkewitsch (Cincinnati, Ohio)

4846:

Derman, Cyrus. A simple allocation problem. Management Sci. 5 (1959), 453–459.

The function $G(a_1, a_2, \cdots, a_k)$ is to be maximized under the condition that the single inequality $\sum_{i=1}^{k} a_i \leqq T$ be fulfilled and $a_i \geqq 0$. If G is non-decreasing in all variables the inequality can be replaced by an equality. The maximum of G is achieved at an interior point if:

$$(\partial G/\partial a_i)|_{a_i=0} \geqq (\partial G/\partial a_j)|_{0<a_j<T}$$

for all i, j.

The method is applied to a problem of allocation under random demand. For the case that the demand distribution is not known the stochastic approximation method of H. Robbins and S. A. Monro [Ann. Math. Statist. 22 (1951), 400–407; MR 13, 144] is suggested.

G. Tintner (Ames, Iowa)

4847:

Fulkerson, D. R. Increasing the capacity of a network: the parametric budget problem. Management Sci. 5 (1959), 472–483.

The paper is concerned with the flow from an origin 0

through a network of links of finite capacity to a destination D. For each link, the cost of providing an additional unit of capacity is known. How should a given budget be spent on the capacity increase of links to achieve the greatest possible increase in the flow from 0 to D that can be handled by the network? It is shown that the labeling procedure previously developed by Ford and Fulkerson [Canad. J. Math. **9** (1957), 210–218; MR **19**, 1244] for the solution of problems of maximal network flow can be modified to furnish the solution of the present problem for all budgets. *W. Prager* (Providence, R.I.)

4848:

Gaver, D. P., Jr. Renewal-theoretic analysis of a two-bin inventory control policy. Naval Res. Logist. Quart. **6** (1959), 141–163.

The author analyses a continuous time inventory problem, in which the demand process is a compound Poisson process, the size of the jumps being exponentially distributed. A general distribution of delivery times is considered, though it is assumed that there is at most one outstanding shipment at any time. An (S, s) type of policy is used and the limiting distribution of stock on hand (which depends on the policy parameters) is computed. This distribution is then used to compute the optimal (S, s) policy in the face of a simple cost structure. The techniques are similar to those used by Karlin in Chap. 15 of *Studies in the mathematical theory of inventory and production* [Stanford Univ. Press, 1958; MR **20** #767] by Arrow, Karlin, and Scarf.

H. Scarf (New Haven, Conn.)

4849:

Brodi, S. M. On an integro-differential equation for systems with τ-waiting. Dopovidi Akad. Nauk Ukraïn. RSR **1959**, 571–573. (Ukrainian. Russian and English summaries)

The author considers the simplest flow of demands on the service of one instrument. The time of service is a random variable having exponential distribution. The demand entering the system is served immediately if the instrument is not engaged and has to wait in line if the instrument is engaged. The demand waits for service no longer than the time τ, after which it is lost. Integro-differential equations for the time of waiting for service are obtained in the paper. *Author's summary*

4850:

★Gillies, Donald B. Solutions to general non-zero-sum games. Contributions to the theory of games, Vol. IV, pp. 47–85. Annals of Mathematics Studies, no. 40. Princeton University Press, Princeton, N.J., 1959. xi+453 pp. $6.00.

The author begins a systematic theory of solutions. The principal result is that a positive fraction of all n-person games are uniquely solvable. Each game whose characteristic function satisfies certain inequalities (the set of such games has interior points in the game space) is shown to have as unique solution the core (the set of undominated imputations). The principal tool in the proof is majorization. α majorizes β means that $\alpha \succ \beta$, and that $\gamma \succ \alpha$ implies $\gamma \succ \beta$. It is proved that the majorized imputations may be ignored in solution theory. More precisely, K is called P-stable if $K = P - \text{dom } K$. If A is the set of all imputations, and A^* the set of unmajorized imputations, then A-stability is equivalent to A^*-stability. This last result is proved for a large class of sets P of generalized imputations, including all compact sets in E_n, and the sets E and $\bar{E}$ defined below. Surprisingly many (but in view of the Kalisch-Nering result on $\aleph_0$-person games in the same Study, of course not all) links in the chain of reasoning depend only upon the irreflexivity of dominance, relating this to Richardson's work [Pacific J. Math. **5** (1955), 551–584; MR **19**, 237]. A corollary of the main result is that a positive fraction of all constant-sum n-person games have solutions. Concerning the four sets in E_n: $\bar{E} = \{\alpha \mid \sum \alpha_i \leqq M\}$, E = boundary of $\bar{E}$, $\bar{A} = \{\alpha \mid \alpha \in \bar{E}$ and $\alpha_i \geqq v(i)\}$, $A = \bar{A} \cap E$, E is essentially as general as $\bar{E}$ in the context of solution theory, and similarly for A and $\bar{A}$. Moreover, E-stable sets are bounded. The author notes also that the restrictions $M = v(I_n)$ and superadditivity of v are irrelevant in this context. The introduction of an $(n+1)$-person "pyramid game" by adjoining an indispensable player shows promise as a general method in solution theory. *J. H. Blau* (Stanford, Calif.)

4851:

★Kemeny, John G. A new approach to n-person games. Contributions to the theory of games, Vol. IV, pp. 397–406. Annals of Mathematics Studies, no. 40. Princeton University Press, Princeton, N.J., 1959. xi+453 pp. $6.00.

The author accepts the characteristic function for zero-sum games, and proposes a framework for supplying additional information concerning bargaining ability and the like. The objective is to remove the wide freedom allowed imputations in solution theory, and to arrive at a single imputation. The players are assigned utility functions f_i, and each coalition divides its payoff evenly relative to the f_i of its members. Knowledge of the characteristic function and of the f_i enables each player to rank all coalitions. If any coalition is the first choice of each of its members, it forms. If not, choose the coalition which minimizes the maximum "sacrifice" of its members. (A member's sacrifice is the difference between his share of his first choice coalition and his share of the given coalition.) The existence of other procedures is indicated. The problem of apportioning losses in the complementary set is not discussed. A number of examples are worked out for the case where each $f_i(x)$ is power of x, leading to division according to fixed ratios. In the 3-person case, unique imputations are obtained with rare exceptions. The theory is not (nor apparently is it intended to be) invariant, even qualitatively, under strategic equivalence. To make it invariant, it would be necessary to vary the f_i as the game varies within an equivalence class.

J. H. Blau (Stanford, Calif.)

BIOLOGY AND SOCIOLOGY

See also 4483, 4486.

4852:

★Bailey, Norman T. J. Statistical methods in biology. John Wiley & Sons, Inc., New York, 1959. ix+200 pp. $4.50.

4853:
Schucker, R. E. A note on the use of triads for paired comparisons. Psychometrika **24** (1959), 273–276.

4854:
Edwards, Allen L. A note on Tryon's measure of reliability. Psychometrika **24** (1959), 257–260.

4855:
Kaiser, Henry F. A note on the Tryon-Kaiser solution for the communalities. Psychometrika **24** (1959), 269–271.

The Tryon-Kaiser solution for the communalities is reviewed. Numerical investigation suggests that the procedure is applicable if and only if the correlation matrix has unique minimum rank communalities.

4856:
Hoffman, Paul J. Generating variables with arbitrary properties. Psychometrika **24** (1959), 265–267.

There are occasions in psychological research where it is desirable to have available sets of variables with arbitrary intercorrelations. A quite simple procedure is described for generating pairs of such variables.

Author's summary

INFORMATION AND COMMUNICATION THEORY

See also 4107, 4489, 4735.

4857:
★Zemanek, Heinz. Elementare Informationstheorie. R. Oldenbourg, Vienna-Munich, 1959. 120 pp. DM 14.20; öS 86.00.

The author considers the area of elementary information theory as encompassing the levels of logical networks, statistical distributions, and statistical relations. The exposition aims to require a minimum of mathematical background of the reader, but does include some consideration of band limited white noise. The first chapter deals with logical networks, essentially the classification of messages expressed as sequences of symbols. The logarithm of the number of messages in the source is introduced as a measure of information. The second chapter treats information as a statistical phenomenon by introducing and considering the probability distribution of the message symbols and groups. Some coding schemes to yield a minimum mean number of symbols per message are described. There is also a discussion of these concepts in relation to spoken language, the nervous system, auditory and visual reception, molecular structures, and germ cells. The third chapter takes up the question of the statistical relation between two sequences of symbols, particularly a transmitted message and the received message. In this chapter appear the notions of channel, capacity of a channel, rate of transmission of information and error correcting codes. Chapter four takes up the transition to continuous procedures and includes a discussion on white noise and pulse code modulation. There is a bibliography of 65 items, a supplement listing binary logarithms to 5 decimal places of the nine digits, powers of ten from 1 to 9, 0.01(.01)0.99, 10(1)999, a table to 5 decimal places of the function $b(p) = -p\mathrm{ld}p - (1-p)\mathrm{ld}(1-p)$ (where ld means logarithm to the base 2) for

$$p = 0.000(.002)0.020(.005)0.050(.010)0.100(.050)0.400(.010)0.450(.005)0.480(.002)0.500,$$

and an index. On page 46 there is a reference to a nonexistent figure 7.3. The definition of TG on page 78 needs a logarithm. Page 85 is particularly plagued by errors. One must agree with the author that a reader with appropriate background in mathematics and statistics would do better to study Shannon's work or others of the rapidly growing special literature.

S. Kullback (Washington, D.C.)

4858:
Ovseevič, I. A.; and Pinsker, M. S. Optimum linear shaping and correction of signal transmitted over a multiple system. Izv. Akad. Nauk SSSR Otd. Tehn. Nauk Energet. Avtomat. **1959**, no. 2, 49–59. (Russian)

The authors consider a two-pole transmission system comprising n linear channels. Channel i ($i = 1, 2, \cdots, n$) consists of a predistortion filter K_i with transmission function $k_i(\omega)$, a linear channel L_i with transmission function $l_i(\omega)$ and additive noise $\{\nu_i(t)\}$, and a receiver M_i with transmission function $m_i(\omega)$. The input process $\{\xi(t)\}$ is applied to $K_1, K_2, \cdots, K_n$, and the output $\{\xi'(t)\}$ is the sum of outputs of $M_1, M_2, \cdots, M_n$. The ξ process is assumed to be stationary, and the ν_i processes are assumed to be stationary and stationarily correlated. The central problem is that of determining the $k_i(\omega)$ and $m_i(\omega)$ which minimize the mean-square error $\sigma^2 = E\{|\xi(t) - \xi'(t)|\}$. The authors solve several special cases of this problem, as well as the problem of maximizing the signal-to-noise ratio at the output of the system. The results are too complicated for reproduction.

L. A. Zadeh (Berkeley, Calif.)

4859:
Bose, R. C.; and Shrikhande, S. S. A note on a result in the theory of code construction. Information and Control **2** (1959), 183–194.

Let $M(n, d; m)$ denote a code of n binary digit messages, having minimum distance d, and consisting of m messages. The main result of this paper is a theorem asserting the coexistence of the maximal codes $M(4t, 2t; 8t)$, $M(4t-1, 2t, 4t)$, the balanced incomplete block design with parameters $v = b = 4t-1$, $r = k = 2t-1$, $\lambda = t-1$, and the Hadamard matrix H_{4t} of order $4t$. The paper thus shows an interesting connection between several fields of mathematics and statistics.

R. W. Hamming (Murray Hill, N.J.)

4860:
Perez, Al'bert. Information theory with an abstract alphabet: generalized forms of McMillan's theorem for the case of discrete and continuous time. Teor. Veroyatnost. i Primenen. **4** (1959), 105–109. (Russian. English summary)

A generalization of McMillan's theorem on the mean convergence of $-n^{-1}\log Pr\{x_1, \cdots, x_n\}$, and one of the analogous two-dimensional results (suitable for source-channel combinations) are discussed very briefly. The

generalizations are to an arbitrary alphabet, and to continuous time (roughly, Feinstein's semi-continuous channel, in the second instance). The paper seems to be largely a résumé of the author's earlier work [Transactions of the first Prague conference on information theory, statistical decision functions, random processes, Liblice, 1956, pp. 183–208, Publishing House Czechoslovak Acad. Sci., Prague, 1957; MR 20 #6325].

V. E. Beneš (Murray Hill, N.J.)

SERVOMECHANISMS AND CONTROL

See also 4107, 4489.

4861:

Kozlov, Yu. M. Calculation of tracking systems with self-regulating parameters. Izv. Akad. Nauk SSSR Otd. Tehn. Nauk Energet. Avtomat. **1959**, no. 2, 65–70.

Consider a linear system characterized by the differential equation

$$\sum_{k=0}^{n} a_k(\rho)x^{(k)}(t) = \sum_{i=0}^{m} b_i(\rho)g^{(i)}(t),$$

in which g is the input, x is the output, and the coefficients a_k $(k=1, 2, \cdots, n)$ and b_i $(i=1, 2, \cdots, n)$ are functions of a parameter $\rho=\rho(t)$ which is a slowly varying function of time. The performance index of the system is identified with the function $F(t)=\sigma_d^2(t)+\sigma_N^2(t)$, where $\sigma_d(t)$ and $\sigma_N^2(t)$ represent, respectively, the dynamic error and mean-square error in tracking a noisy input. By employing the technique of time-dependent frequency response functions, the author obtains an equation satisfied by a $\rho(t)$ which minimizes $F(t)$. The results are approximate in nature, with the accuracy depending on the rate of variation of $\rho(t)$ compared with that of the impulse response of the system. *L. A. Zadeh* (Berkeley, Calif.)

4862:

Sawaragi, Yoshikazu; and Sugai, Narinobu. A graphical method for the evaluation of the statistical responses of non-linear control systems to a stationary random signal considering random noise. Mem. Fac. Engrg. Kyoto Univ. **21** (1959), 192–213.

The principal contents of this paper, other than some examples illustrating the use of the theory, have already appeared in a series of three articles by the first named author and Y. Sunahara [Tech. Rep. Engrg. Res. Inst. Kyoto Univ. **8** (1958), 95–126, 195–218; **9** (1959), 77–96; MR **20** #793; **21** #4067, #4068]. Apparently, the present paper is intended as a shortened and simplified version of the theory for the practical use of engineers.

L. A. MacColl (New York, N.Y.)

4863:

Amara, R. C. Application of matrix methods to the linear least squares synthesis of multivariable systems. J. Franklin Inst. **268** (1959), 1–16.

The chief purpose of this paper is the demonstration of the application of matrix methods to linear systems, subject to stationary stochastic inputs, which feature a multiplicity of inputs and outputs. It is desired to find the optimum transfer-function matrix when the criterion is the minimization of the sum of the mean-square errors between the set of actual outputs and a set of desired outputs. The problem is formulated and the derivation of the minimization equations is accomplished without matrices; the results are then expressed in matrix form. The paper continues with the explicit matrix solution of the minimization equations both for the filter problem (in which the only restriction on the linear transfer function matrix is that of physical realizability) and the control problem (in which additional restrictions on the over-all performance of the system are included). An appendix illustrates the method of solution of a simple filter problem with 2 inputs and 2 outputs. *P. S. Dwyer* (Ann Arbor, Mich.)

4864:

Roitenberg, Ia. N. On the accumulation of disturbances in transient linear impulsive systems. J. Appl. Math. Mech. **22** (1958), 741–744 (534–536 Prikl. Mat. Meh.).

Consider a system characterized by the equations

$$(*) \quad y_k(t+\tau)+\sum_{l=1}^{n} a_{kl}(t)y_l(t) = x_k(t) \quad (k=1, \cdots, n),$$

where the $x_k(t)$ and $y_k(t)$ represent inputs and outputs, respectively, and the $a_{kl}(t)$ are given functions of time. Let $|x_k(t)| \leq L_k$ $(k=1, \cdots, n)$, where the L_k are specified constants. The author obtains an upper bound for $|y_s(t_1)|$, where $1 \leq s \leq n$ and t_1 is a fixed value of t, in terms of solutions of the adjoint of (*) and initial values of $y_k(t)$ $(k=1, \cdots, n)$ in the interval $(0, \tau)$.

L. A. Zadeh (Berkeley, Calif.)

4865:

★Айзерман, М. А. Лекции по теории автоматического регулирования. [Aizerman, M. A. Lectures on the theory of automatic regulation.] 2nd ed. enlarged and revised. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1958. 520 pp. 15.75 rubles.

The second edition [see MR **18**, 981 for review of first, 1956, edition] of this excellent compact text on automatic control includes new material on extremum regulators (optimalizing servomechanisms), structural stability, statistical methods and, in particular, exact methods of analysis of nonlinear systems with piecewise linear characteristics. The latter is based on the recent work of the author and F. R. Gantmacher [Avtomat. i. Telemeh. **18** (1957), 97–110; MR **21** #1922].

L. A. Zadeh (Berkeley, Calif.)

4866:

Boguslavskiĭ, I. A. On a class of nonstationary random process filtering. Avtomat. i Telemeh. **20** (1959), 708–720. (Russian. English summary)

The author solves the following variant of the filtration problem: Given a random process $x(t)$ which is the sum of a noise process $n(t)$ and a signal process $s(t)$ satisfying the differential equation

$$\dot{s}(t) = a(t)s(t)+b(t)+\sum_{k=1}^{n} c_k f_k(t),$$

where the $f_k(t)$ are known functions, $a(t)$ and $b(t)$ are not known in advance but are continuously measurable during filtration, and the c_k are unknown numbers which are fixed for any realization of the process $s(t)$. Find the linear

time-varying filter which for input $x(t)$ has for its average output the function

$$z(t) = s(t) + \sum_{k=1}^{n} c_k g_k(t),$$

where the $g_k(t)$ are known functions. The method is applied to the problem of finding angular coordinates of an aircraft tracked by a radar located on a moving vessel.

R. A. Silverman (New York, N.Y.)

4867:

Puig Adam, P. An electrical device for solving problems of formal logic. Rev. Acad. Ci. Madrid **53** (1959), 11–31. (Spanish)

Description of a simple network illustrating propositional logic.

4868:

Lowenschuss, O. Restoring organs in redundant automata. Information and Control **2** (1959), 113–136.

This paper elaborates on a few of the ideas from a paper by von Neumann [*Automata studies*, pp. 43–98, Princeton Univ. Press, 1956; MR **17**, 1040], by permitting the unreliable components to be n-valued rather than 2-valued. The restoring organs used by von Neumann in improving reliability are also modified here by permitting them not to be majority organs, but instead to be composite organs made up of several simpler components.

E. F. Moore (Murray Hill, N.J.)

4869:

Netherwood, Douglas B. Logic matrices and the truth function problem. J. Assoc. Comput. Mach. **6** (1959), 405–414.

Two matrices of 0's and 1's are of the same symmetry type if one can be transformed into the other by permuting rows, permuting columns, and complementing columns. Among all matrices of the same type as a given one, this paper attempts to describe a procedure for picking one which is to be considered canonical. The author tries to give a procedure better suited for machine computation than the enumeration of all matrices of the same type as the given one. He has not given a well-defined procedure, because of the vagueness of certain statements he makes, and also because he makes use of his theorem 1, of which the following matrix is a counterexample:

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

E. F. Moore (Murray Hill, N.J.)

4870:

Kochen, Manfred. Circle networks of probabilistic transducers. Information and Control **2** (1959), 168–182.

This paper investigates the behavior of circular networks, made up of devices, each of which takes on the state 0 or the state 1 at each integer-valued time. The probability that a given device will be in state 0 at time $t+1$ depends only on the states at time t of the same device and of one immediately neighboring device on each side. If none of the probabilities defining the behavior is 0 or 1, in the limit the behavior of each device becomes independent of the initial state of all of the devices. Several theorems analogous to this, but more specific in various ways, are correctly proved in this paper. But in the first two sentences of his abstract the author claims that it is shown that each device will eventually have behavior independent of its neighbors. Counterexamples can easily show this statement is not true, and apparently his claim is merely a vague and mistaken generalization of his valid results.

E. F. Moore (Murray Hill, N.J.)

4871:

Lee, C. Y. Representation of switching circuits by binary-decision programs. Bell System Tech. J. **38** (1959), 985–999.

Consider a two-terminal switching circuit σ with N two-state variables, where N is some positive integer. A binary-decision program for σ is a sequence of ordered triples $(x_i, A, B)_k$, where k is an integer which is the index for the sequence and x_i $(i=1, 2, \cdots, N)$ is a two-state variable of the switching circuit. Each ordered triple is called an instruction and the index k is called the address of that instruction. If $x_i=0$, then A is the address of the next instruction. If $x_i=1$, then B is the address of the next instruction. In addition, there are two bizarre addresses θ and I, which denote the condition of an open-circuit and short-circuit, respectively, for σ.

In terms of three inherently non-algebraic rules, the author treats two specific examples to demonstrate the procedure of going from a logical statement to a binary-decision program and finally to a relay-contact switching circuit. The author says "Binary-decision programs ... are more suited for representing circuits with a large number of transfers"; however, the reviewer points out that the more conventional symmetric switching functions are useful in this situation, also.

Next, the author considers some aspects of a measure of complexity associated with switching functions and develops some pertinent theorems.

{The reviewer believes that there is much to be learned about the nature of conventional switching circuit theory, if only in the light of the interesting approaches by means of decision-programs and recursive function theory.}

A. A. Mullin (Urbana, Ill.)

4872:

Stewart, D. J. A notation for logical nets. Cybernetica **2** (1959), 59–69.

This paper presents a notation for logical nets somewhat more general than those considered by McCulloch and Pitts, von Neumann, and Kleene. The innovation consists in associating with each inner element of a net an inhibition number h such that if x is the number of excitatory inputs and y is the number of inhibitory inputs of the element, then the element fires if and only if $x-y \geq h$. Other notations are shown to be special cases of this, and examples of net diagrams which realize certain logical functions are given in this notation. In addition an example is provided of a simple net which displays an effect of learning.

E. J. Cogan (Bronxville, N.Y.)

HISTORY AND BIOGRAPHY

4873:

★**Smith, David Eugene. A source book in mathematics.** 2 vols. Dover Publications, Inc., New York, 1959. xii+311+xiii+395 pp. Paperbound. $1.85 per vol.; $3.50 the set.

Unaltered republication of the 1st (one-volume) ed. [McGraw-Hill, New York, 1929]; the present edition is separated into number theory and algebra (vol. 1), and geometry, probability, calculus and quaternions (vol. 2).

4874:

Pinl, M. "100 Jahre Differentialgeometrie." Math.-Phys. Semesterber. **5** (1956), 34–48.

4875:

Godeaux, Lucien. Les recherches de géométrie dans ces dernières années en Belgique. Mém. Publ. Soc. Sci. Arts Lett. Hainaut **73** (1959), 55–67.

4876:

Kulczycki, S. On Riemann's habilitational address. Wiadom. Mat. (2) **1** (1955/56), 180–193. (Polish)

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